

17 May 2005

Discrete vs. Smooth Connections

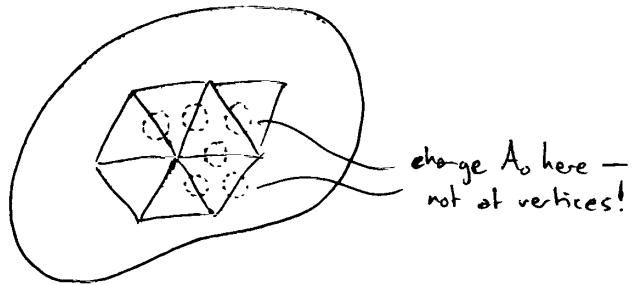
Given a Lie gp. G & a principal G -bundle $P \rightarrow M$, we get a space $A(P)$ of smooth connections on P . Given a triangulation Δ of M we get a space of "discrete connections" $A(\Delta)$. Given a trivialization of P over the vertices of Δ (which always exists) we get a map

$$\wedge : A(P) \rightarrow A(\Delta)$$

In fact:

Thm: If G is connected, $\wedge : A(P) \rightarrow A(\Delta)$ is onto.

Proof Sketch:

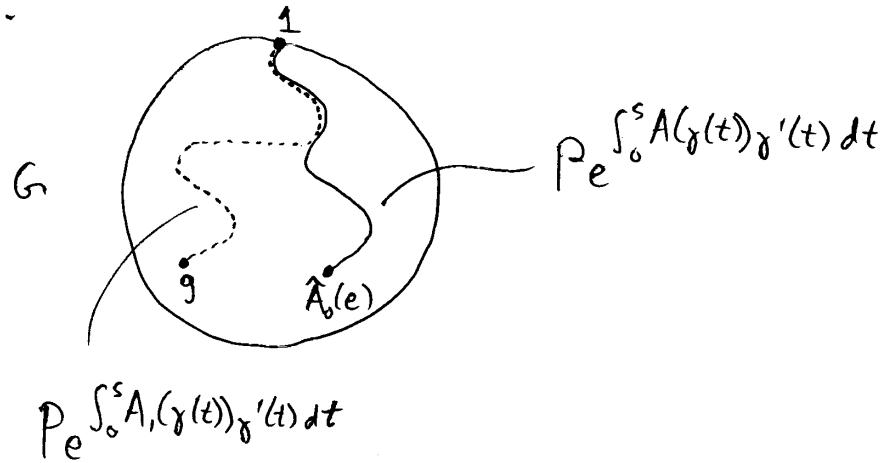


Pick a connection A_0 on P ; \hat{A}_0 will assign a group elt to each edge:

$$\hat{A}_0(e) = P e^{\int_0^1 A_0(\gamma(t))(\gamma'(t)) dt} \in G$$

where $\gamma : [0,1] \rightarrow M$ runs along the edge e , & " P " denotes path-ordered exponential, i.e. the solution of the differential equation describing parallel transport. By changing A_0 in a small open ball centered at the midpoint of the edge e , we can get a new conn.

A_i , that has $\hat{A}_i(e)$ being any desired group element, since G is connected.



Repeating this for each edge, we get connections A_0, A_1, \dots, A_n where A_n has desired holonomy along every edge. ■

We also have two kinds of gauge transformations:

- A smooth gauge transformation is an automorphism of our principal G -bundle P . These form a group $\mathcal{G}(P)$
- A discrete gauge transformation is a map $g: V \rightarrow G$ where V is the set of vertices of Δ . These form a group $\mathcal{G}(\Delta)$.

If we trivialize P at the vertices of Δ , each fiber P_v ($v \in V$) becomes identified with G & so any smooth gauge transformation

$$\alpha: P \rightarrow P$$

gives

$$\begin{array}{ccc} P_v & \xrightarrow{\alpha} & P_v \\ \downarrow & & \downarrow \\ G & \xrightarrow{f} & G \end{array}$$

where the G -torsor isomorphism $f: G \rightarrow G$ must be of the form

$$f(h) = gh \quad \forall h \in H$$

for some unique $g \in G$. So we get a discrete gauge transformation

$$\begin{aligned} g: V &\rightarrow G \\ v &\mapsto g_v \end{aligned}$$

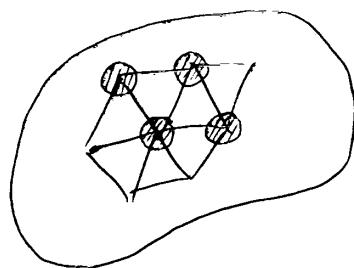
So, we obtain

$$\begin{aligned} {}^{\wedge}: \mathcal{G}(P) &\rightarrow \mathcal{G}(\Delta) \\ \alpha &\longmapsto g \end{aligned}$$

This is not 1-1, since we can easily find two different gauge transformations that agree at all $v \in V \subseteq M$.

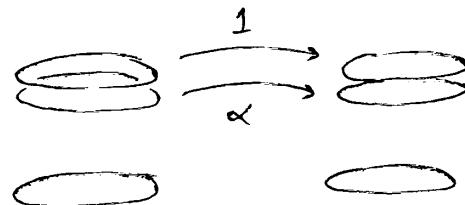
Thm: If G is connected, ${}^{\wedge}: \mathcal{G}(P) \rightarrow \mathcal{G}(\Delta)$ is onto.

Proof Sketch:

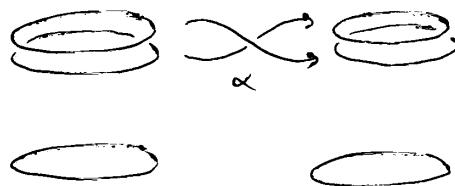


By changing α only in small open balls centered at the vertices, we can make $\alpha: P_v \rightarrow P_v$ be whatever we want, while still keeping it smooth (since G is connected). So, we can find some $\hat{\alpha} \in \mathcal{G}(P)$ that makes $\hat{\alpha} \in \mathcal{G}(\Delta)$ be whatever we want. \blacksquare

\wedge is sometimes onto even when G is not connected



$$\hat{\alpha}(v) = 1 \in \mathbb{Z}_2$$

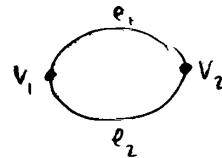


$$\hat{\alpha}(v) = -1 \in \mathbb{Z}_2$$

Both for the trivial & the nontrivial \mathbb{Z}_2 bundle over S^1 ,

$$\wedge : \mathcal{G}(P) \longrightarrow \mathcal{G}(\Delta)$$

is onto for the triangulation Δ with one vertex and one edge. For another triangulation, say



the map \wedge will not be onto. This is the typical situation (when G is connected).

Next note that $\mathcal{G}(P)$ acts on $A(P)$, and once upon a time we saw that $\mathcal{G}(\Delta)$ acts on $A(\Delta)$. In fact

$$\hat{\alpha} \hat{A} = \hat{\alpha} \hat{A}$$

i.e. the \wedge maps preserve the group action.

i.e.

$$g(P) \times A(P) \longrightarrow A(P)$$

$$\begin{array}{c} \wedge \\ \downarrow \\ \wedge \end{array}$$

$$\begin{array}{c} \wedge \\ \downarrow \\ \wedge \end{array}$$

$$g(\Delta) \times A(\Delta) \longrightarrow A(\Delta)$$

commutes. So we get a map

$$\wedge : A(P)/g(P) \longrightarrow A(\Delta)/g(\Delta)$$

i.e.

$$[A] = [A'] \quad A, A' \in A(P)$$

$$\downarrow$$

$$A' = \alpha A$$

$$\downarrow$$

$$\hat{A}' = \hat{\alpha} \hat{A}$$

$$\downarrow$$

$$\hat{A}' = \hat{\alpha} \hat{A}$$

$$\downarrow$$

$$[\hat{A}] = [\hat{A}']$$

Note this \wedge is onto, since $\wedge : A(P) \rightarrow A(\Delta)$ is onto.

Alas, it's still not 1-1. But, it becomes so if we consider flat connections:

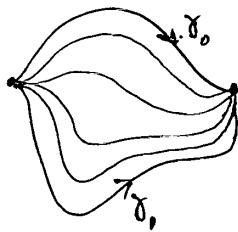
- A flat smooth connection A is one for which the curvature F is zero.
- A flat discrete connection is one such that the holonomy around any triangle is 1:



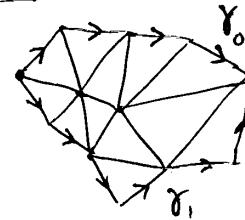
$$A(e_1)A(e_2)A(e_3) = 1.$$

In both cases, flatness is equivalent to the holonomy along any path being invariant under homotopy:

smooth:



discrete:



$$H(A, \gamma_0) = H(A, \gamma_1)$$

$$A(\gamma_0) = A(\gamma_1)$$

19 May 2005

Moduli Space of Flat Connections & Flat Bundles

We've seen that given a smooth manifold M & a Lie group G , we can set up:

- a theory of smooth connections starting from any principal G -bundle $P \rightarrow M$:

- $\mathcal{A}(P)$ = smooth connections on P
- $\mathcal{G}(P)$ = smooth gauge transformations of P
- $\mathcal{A}_0(P)$ = flat smooth connections on P .

- a theory of discrete connections starting from a triangulation Δ of M :

- $\mathcal{A}(\Delta)$
- $\mathcal{G}(\Delta)$
- $\mathcal{A}_0(\Delta)$

We also have maps

$$\wedge : A(P) \longrightarrow A(\Delta)$$

$$\wedge : g(P) \longrightarrow g(\Delta)$$

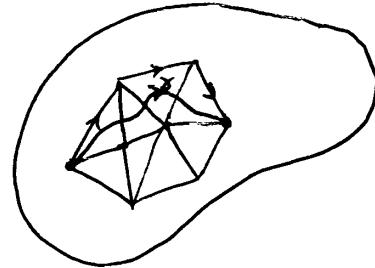
$$\wedge : A_o(P) \longrightarrow A_o(\Delta)$$

In both smooth & discrete contexts, gauge transformations act on connections & preserve the subspace of flat ones, so we can form $A_o(P)/g(P)$ & $A_o(\Delta)/g(\Delta)$, and in fact we get

$$\wedge : A_o(P)/g(P) \longrightarrow A_o(\Delta)/g(\Delta)$$

Thm: $\wedge : A_o(P)/g(P) \longrightarrow A_o(\Delta)/g(\Delta)$ is 1-1.

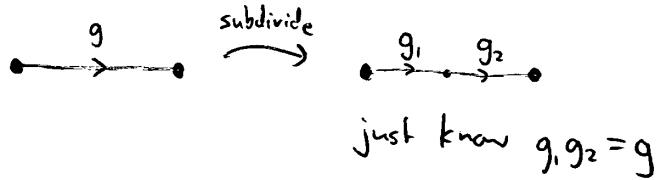
Proof Sketch: If $A \in A_o(P)$ and $v, v' \in V$ (the set of vertices of Δ) then A determines parallel transport along all smooth paths γ from v to v' . \hat{A} , on the other hand, only determines parallel transport along edge paths, like S . Note: edge paths are actually smooth if we parameterize them nicely at the vertices (all derivatives zero!). Since every smooth path from v to v' is homotopic to an edge path \hat{A} determines parallel transport along all smooth paths from v to v' .



Note: we don't have that

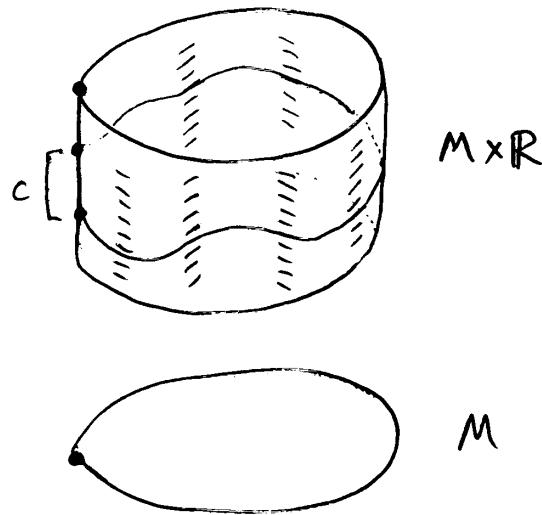
$$\wedge : A_o(P) \longrightarrow A_o(\Delta)$$

is 1-1, since knowing parallel transport along an edge doesn't determine it along a piece of an edge:



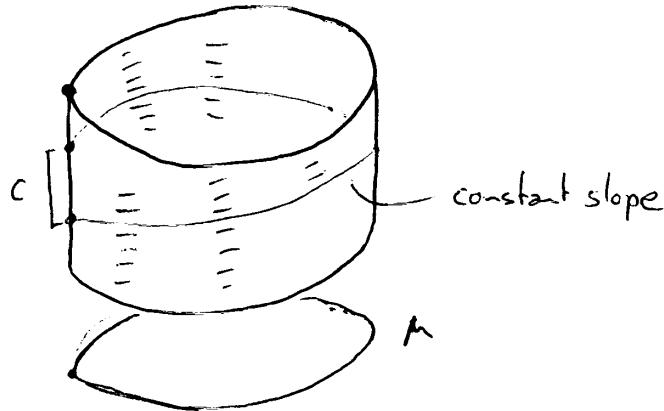
So $\hat{A} \in A_o(\Delta)$ does not determine $A \in A_o(P)$. But if does determine $[A] \in A_o(P)/G(P)$ — this says $\wedge : A_o(P)/G(P) \rightarrow A_o(\Delta)/G(\Delta)$ is 1-1.

So we need: if we know what holonomy A assigns to paths between vertices, then we know $[A] \in A_o(P)/G(P)$.



Example — all connections on a 1-manifold are flat; take $G = \mathbb{R}$, $M = S^1$, pick a connection A (flat!) and check:

does the holonomy of A along all paths between vertices determine $[A] \in A_0(P)/g(P)$. Yes! Reason: we can gauge transform A so that it looks like this:



This connection is clearly determined by $c \in \mathbb{R}$, the holonomy around the circle. So $[A]$ is determined. In general we'd prove the final claim by showing that any flat connection with prescribed holonomies along all paths between vertices can be gauge transformed to some standard form. ■

Now: is

$$\wedge : A_0(P)/g(P) \rightarrow A_0(\Delta)/g(\Delta)$$

onto? Sometimes! ...

Thm: Suppose M is a compact oriented 2-manifold and $G = U(1)$. Principal $U(1)$ bundles $P \rightarrow M$ are classified up to isomorphism by an integer called the first Chern number:

$$c_1(P) = \frac{1}{2\pi} \int_M F$$

where F is the curvature of any connection A on P (thinking of F as a 2-form). If $c_1(P) \neq 0$, $A_0(P) = \emptyset$ and $\wedge: A_0(P)/g(P) \rightarrow A_0(\Delta)/g(\Delta)$ is not onto. If $c_1(P) = 0$ then \wedge is 1-1 and onto.