

24 May 2005

## The Moduli Space of Flat Bundles

Recall: if  $P \rightarrow M$  is a principal  $G$ -bundle &  $\Delta$  is a triangulation we get

$$\wedge: A_0(P)/g(P) \longrightarrow A_0(\Delta)/g(\Delta)$$

where  $A_0(P)/g(P)$  is called the moduli space of flat connections on  $P$ , where "moduli space" just means a space of equivalence classes of some geometrical structure (connection, complex structure, ...). Similarly  $A_0(\Delta)/g(\Delta)$  is called the moduli space of flat bundles, for a reason soon to be revealed.

We've seen that  $\wedge$  is always 1-1, but not always onto. However:

Thm: We can define

$$\wedge: \coprod_{\substack{[P] \text{ iso classes} \\ \text{of principal } G\text{-} \\ \text{bundles over } M}} A_0(P)/g(P) \longrightarrow A_0(\Delta)/g(\Delta)$$

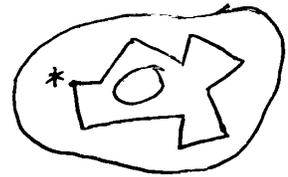
to equal the previously defined  $\wedge$  map on each  $A_0(P)/g(P)$ ; this map is onto.

If this is 1-1 also, then  $A_0(\Delta)/g(\Delta)$  deserves its name, the moduli space of flat bundles, since its points are the same as: isomorphism classes of principal  $G$ -bundles  $P \rightarrow M$  with flat connection.

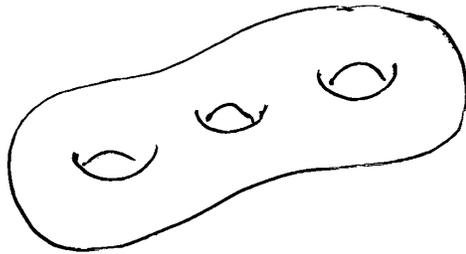
Last quarter we saw

$$A_0(\Delta)/g(\Delta) \cong \text{hom}(\pi_1(M), G)/G$$

if  $M$  is connected, since a flat <sup>(discrete)</sup> connection assigns a group element to each loop based at  $* \in M$ , which does not change when we do a homotopy, so we get a homomorphism from  $\pi_1(M)$  to  $G$ ; applying a gauge transformation conjugates this by an element of  $G$ .



Now assume  $M$  is a compact oriented connected 2-manifold:



Last time we saw:

Thm: If  $G = U(1)$ , principal  $G$ -bundles  $P \rightarrow M$  are classified by

$$c_1(P) = \frac{1}{2\pi} \int_M F$$

where  $F$  is the curvature of any connection. Here  $A_0(P)/g(P)$  is empty unless  $c_1(P) = 0$ ; if  $c_1(P) = 0$   $P$  is trivial and then

$$\wedge: A_0(P)/g(P) \longrightarrow A_0(\Delta)/g(\Delta)$$

is 1-1 & onto, so

$$\begin{aligned}
 A_0(P)/g(P) &\cong A_0(\Delta)/g(\Delta) \\
 &\cong \text{hom}(\pi_1(M), U(1))/U(1) \\
 &\cong \text{hom}(\pi_1(M), U(1)) && \left. \begin{array}{l} \swarrow U(1)\text{-abelian.} \\ \searrow U(1)\text{abelian} \end{array} \right\} \\
 &\cong \text{hom}(H_1(M), U(1))
 \end{aligned}$$

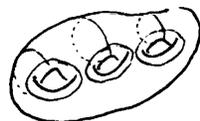
since the first homology group  $H_1(M)$  is the abelianization of  $\pi_1(M)$ , so every homo. to an abelian gp factors through  $H_1(M)$ . In fact

$$\text{hom}(H_1(M), U(1)) \cong H^1(M, U(1))$$

— the first cohomology group of  $M$  with coefficients in  $U(1)$ , thanks to the "universal coefficient theorem" (Ext vanishes).

In fact: if  $M$  has genus  $g$

$$\begin{aligned}
 H^1(M, U(1)) &\cong \text{hom}(\mathbb{Z}^{2g}, U(1)) \\
 &\cong U(1)^{2g}
 \end{aligned}$$



Thm: If  $G = SU(n)$  and  $M$  is a compact connected oriented 2-manifold, every principal  $G$ -bundle  $P \rightarrow M$  is trivial, and

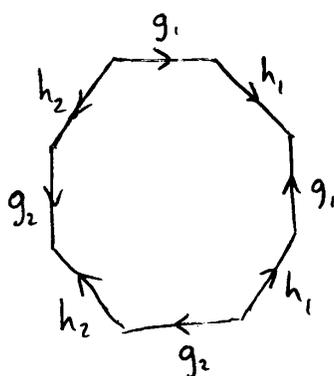
$$\wedge: A_0(P)/g(P) \longrightarrow A_0(\Delta)/g(\Delta)$$

is 1-1 & onto, so

$$A_0(P)/g(P) \cong \text{hom}(\pi_1(M), SU(n))/SU(n)$$

Witten's work on 2d gauge theory let people understand  $\text{hom}(\pi_1(M), \text{SU}(n))/\text{SU}(n)$  in excruciating detail.

Thm: If  $G = \text{SO}(3)$  (and  $M$  is still a surface) and  $P \rightarrow M$  is any principal  $G$ -bundle, there exists a flat connection on it, say  $A$ . Then  $\hat{A}$  assigns an element of  $\text{SO}(3)$  to each of the generators of  $\pi_1(M)$ :



( $g=2$ )

say  $g_i, h_i \in \text{SO}(3)$  ( $i=1, \dots, g$ ) and they satisfy

$$(g_1 h_1 g_1^{-1} h_1^{-1}) (g_2 h_2 g_2^{-1} h_2^{-1}) \dots = 1.$$

Since there's a 2-1 homomorphism

$$p: \text{SU}(2) \longrightarrow \text{SO}(3)$$

we can pick  $\hat{g}_i, \hat{h}_i \in \text{SU}(2)$  with

$$p(\hat{g}_i) = g_i \quad p(\hat{h}_i) = h_i$$

and we know

$$(\hat{g}_1 \hat{h}_1 \hat{g}_1^{-1} \hat{h}_1^{-1}) (\hat{g}_2 \hat{h}_2 \hat{g}_2^{-1} \hat{h}_2^{-1}) \dots = \pm 1$$

since  $p$  of it is 1. This number  $\pm 1$  turns out

to be independent of the choice of  $A$  & choice of  $\hat{g}_i, \hat{h}_i$ . It's called the second Stiefel-Whitney class of  $P$ . This number completely classifies  $P$  up to isomorphism: if it's  $1$ ,  $P$  is trivial; if it's  $-1$ ,  $P$  is the unique (up to iso) such nontrivial bundle. Moreover:

$$\begin{aligned} \wedge : A_0(P_{\text{triv}})/g(P_{\text{triv}}) \sqcup A_0(P_{\text{nontriv}})/g(P_{\text{nontriv}}) &\longrightarrow A_0(\Delta)/g(\Delta) \\ &\parallel \\ &\text{hom}(\pi_1(M), SO(3))/SO(3) \end{aligned}$$

is 1-1 and onto. **■**

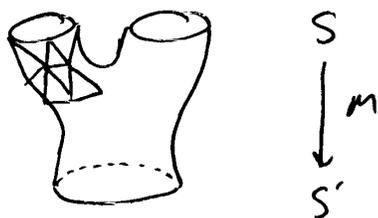
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## From the 2D Dijkgraaf-Witten Model to 2D EF Theory

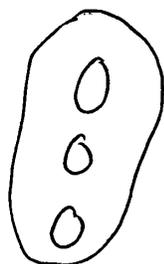
The Dijkgraaf-Witten model is a TQFT based on a finite group  $G$ , in which the partition function

$$\tilde{Z}(M) : \tilde{Z}(S) \longrightarrow \tilde{Z}(S')$$



is calculated as a sum over discrete flat connections on a triangulation of  $M$ , which for

$$M : \emptyset \longrightarrow \emptyset$$



a  $g$ -holed torus

amounted to

$$\tilde{Z}(M) : \mathbb{C} \longrightarrow \mathbb{C}$$

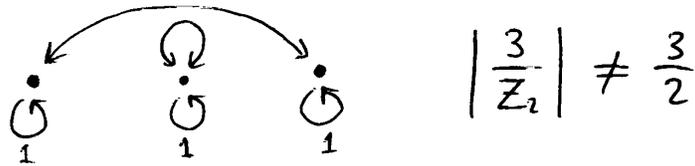
i.e. a number:

$$\tilde{Z}(M) = \frac{|A_0(\Delta_M)|}{|g(\Delta_M)|} \text{ times a factor involving } \chi(M) = 2 - 2g$$

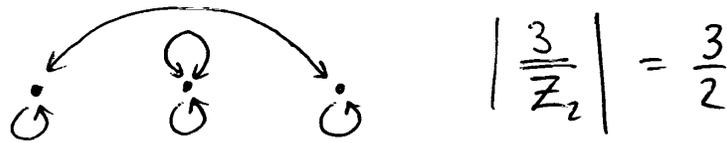
where  $A_0(\Delta_M)$  is the set of discrete flat connections on  $M$  (equipped with some triangulation  $\Delta_M$ ) &  $g(\Delta_M)$  is the

set of discrete gauge transformations on  $M$ .

Cardinality of sets behaves badly under division:



Cardinality of groupoids behaves better



So we should really form a groupoid  $A_0(\Delta_M) // G(\Delta_M)$  with flat connections as objects & gauge transformations  $g: A \rightarrow A'$  as morphisms  $g: A \rightarrow A'$ , and get

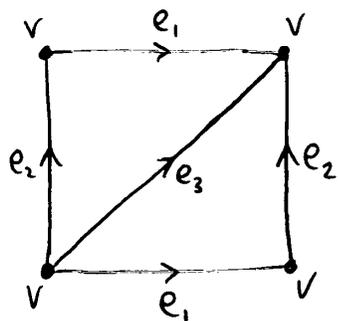
$$\tilde{Z}(M) = \left| \frac{A_0(\Delta_M)}{G(\Delta_M)} \right|$$

Just as the ordinary quotient  $A_0(\Delta_M) / G(\Delta_M)$  is called the moduli space of flat bundles,  $A_0(\Delta_M) // G(\Delta_M)$  is called the moduli stack of flat bundles.

Now suppose  $G$  is a compact Lie group. In "EF theory" we try to generalize everything about the Dijkgraaf-Witten model to this case, using integration with respect to (normalized) Haar measure as a

substitute for summation. We could hope that  $\tilde{Z}(M)$  is given by the same sort of formula as in the finite case. We know what the set  $A_0(\Delta_n)$  & the group  $g(\Delta_n)$  are like — now they're compact topological spaces — and we know how  $g(\Delta_n)$  acts on  $A_0(\Delta_n)$ . We even know painfully explicit descriptions of the moduli space of flat bundles  $A_0(\Delta_n)/g(\Delta_n)$ , for  $G = U(1), SU(n), SO(n)$ . But what about the moduli stack  $A_0(\Delta_n)//g(\Delta_n)$ ? It's a well-defined groupoid — but it has infinitely many isomorphism classes of objects, so we should really find a natural measure on  $A_0(\Delta_n)//g(\Delta_n)$  & integrate it to get a number. But beware:  $A_0(\Delta_n)//g(\Delta_n)$  is a groupoid, so we'd need to understand measures on groupoids! People barely understand this topic, but they do certainly understand a natural measure on  $A_0(\Delta_n)/g(\Delta_n)$ . This isn't quite good enough, but it could be good enough for government work if the reducible connections — the ones that are preserved by more gauge transformations than the rest — are somehow "negligible", i.e. a "set of measure zero." It's hard to tell unless we look more closely...

An example:  $M = T^2$   
 $G = SO(3)$



$$E = \{e_1, e_2, e_3\}$$

$$V = \{v\}$$

$$A(\Delta_M) = SO(3)^E \cong SO(3)^3$$

$$\begin{aligned} A_0(\Delta_M) &= \{(g_1, g_2, g_3) : g_3 = g_1 g_2 = g_2 g_1\} \\ &= \{(g_1, g_2) : g_1 g_2 = g_2 g_1\} \\ &= \text{hom}(\pi_1(M), SO(3)) \end{aligned}$$

$$g(\Delta_M) = SO(3)^V \cong SO(3) \ni h$$

and this group acts on  $(g_1, g_2) \in A_0(\Delta_M)$  by conjugation:

$$h : (g_1, g_2) \longmapsto (h g_1 h^{-1}, h g_2 h^{-1})$$

The moduli space of flat bundles is

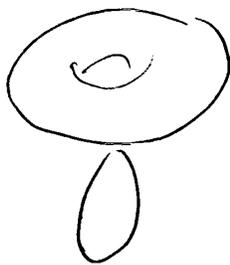
$$\frac{A_0(\Delta_M)}{g(M)} = \frac{\{(g_1, g_2) \in SO(3)^2 : g_1 g_2 = g_2 g_1\}}{\{(g_1, g_2) \sim (h g_1 h^{-1}, h g_2 h^{-1})\}}$$

Note:  $(g_1, g_2) \in A_0(\Delta_M)$  consists of 2 rotations around the same axis, almost always — but beware:

- 1) the identity is a rotation about every axis
- 2)  $180^\circ$  rotations commute if their axes are at  $90^\circ$ .

Conjugating a rotation around axis  $v \in \mathbb{R}^3$  by a rotation  $h$  gives a rotation around axis  $hv$  - a rotated axis.

So barring the exceptional cases, a point in the moduli space is a pair of "angles"  $(\theta_1, \theta_2) \in SO(2)^2$  modulo  $(\theta_1, \theta_2) \sim (-\theta_1, -\theta_2)$ . These points correspond to nonreducible flat connections; some of the annoying special cases include the reducible connections.



2d space of nonreducible  
elts in moduli space

1d space of annoying special cases.