

31 May 2005

The 2D Dijkgraaf-Witten model & EF Theory

If M is a compact oriented 2-manifold with triangulation Δ_M and G is a compact Lie group, we expect that 2d BF theory gives

$$\tilde{Z}(M) = \text{measure of } \frac{A_0(\Delta_M)}{g(\Delta_M)}$$

- the measure of the "moduli stack of flat G -bundles over M ". Until we figure out what a measure on a groupoid should be like, we can wimp out & try

$$\tilde{Z}(M) = \text{measure of } \frac{A_0(\Delta_M)}{g(\Delta_M)}$$

- hoping that reducible connections don't count for much.
There's still the question: which measure?

Thm: There's a natural symplectic structure ω on the open dense subset of $\frac{A_0(\Delta_M)}{g(\Delta_M)}$ consisting of irreducible connections

- a $2k$ -dimensional manifold. Thus

$$\omega^k = \underbrace{\omega \wedge \cdots \wedge \omega}_{k \text{ times}}$$

is a volume form on this open dense subset.

Proof Sketch: We only get our nondegenerate closed 2-form ω thanks to the fact that M is 2-dimensional. ω should eat two tangent vectors

$$u, v \in T_x \frac{A(\Delta_n)}{g(\Delta_n)}$$

& give a number $\omega(u, v)$, whenever $x = [A_0]$ for some irreducible flat connection. I'll just give a formula for $\omega(\delta A, \delta A')$ where

$$\delta A, \delta A' \in T_{A_0} A_0(P)$$

where P is some principal G -bundle $P \rightarrow M$. Note: $\delta A, \delta A' \in \Omega^1(M, \text{Ad } P)$, so we can define

$$\omega(\delta A, \delta A') = \int_M \text{tr}(\delta A \wedge \delta A')$$

where $\text{tr}(\delta A \wedge \delta A')$ is defined by wedge product of 1-form parts, trace of matrix product of g -valued parts (thinking ~~locally~~ of $\text{Ad}(P)$ -val. form as locally a g -val differential form and thinking of g as a Lie algebra of matrices.) This uses the fact that M is 2-dimensional!
Need to check ω is nondegenerate. \blacksquare

Given this, we could try to calculate $\tilde{Z}(M)$ as

$$\int_{\{[\text{irreducible conn.}]\}} \omega^k$$

Or: we could copy what worked in Dijkgraaf Witten model, instead of just generalizing the end result.

2d DW

G finite group

$$\rightarrow A = \mathbb{C}[G] = \{\psi : G \rightarrow \mathbb{C}\}$$



$$m : A \otimes A \rightarrow A$$

given by

$$\delta_g * \delta_h = \delta_{gh}$$

$$\begin{aligned} \text{or } (\psi * \varphi)(g) &= \sum_{\substack{h, h' \in G \\ \text{st. } hh' = g}} \psi(h)\varphi(h') \\ &= \sum_{h \in G} \psi(h)\varphi(h^{-1}g) \end{aligned}$$

2d EF

G compact Lie group

$$\rightarrow A = L^2(G, \mu)$$

(Haar measure)

$$= \{\psi : G \rightarrow \mathbb{C} : \int |\psi|^2 d\mu < \infty\}$$



$$m : A \otimes A \rightarrow A$$

is convolution:

$$\begin{aligned} (\psi * \varphi)(g) &= \int \psi(h)\varphi(h^{-1}g) d\mu(h) \\ (\text{Note: } \int |\psi|^2 < \infty \text{ & } \int |\varphi|^2 < \infty \\ \Rightarrow \int |\psi * \varphi|^2 < \infty.) \end{aligned}$$

$$\begin{array}{c} \diagdown \\ \square \\ \diagup \end{array} = \begin{array}{c} \diagup \\ \square \\ \diagdown \end{array} \quad A \text{ is associative}$$

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$$\begin{array}{c} \downarrow \\ \triangle \\ \downarrow \end{array} = \begin{array}{c} \rightarrow \\ \triangle \\ \downarrow \end{array} \quad \text{comes from:}$$

A is semisimple -
 $g(a,b) = \text{tr}(L_a L_b)$
 is nondegenerate, where
 $L_a : A \rightarrow A$
 $x \mapsto ax$

Alas,

$$g(a,b) = \text{tr}(L_a L_b)$$

is ill-defined: A is infinite dimensional & $\text{tr}(L_a L_b)$ doesn't always converge - e.g. if $a=b=1$ we get $\dim(A) = \infty$.

But all is not lost! We can "smooth out"

$$m: L^2(G) \otimes L^2(G) \longrightarrow L^2(G)$$

$$\psi \otimes \varphi \longmapsto \psi * \varphi$$

to ensure that $\text{tr}(L_a L_b)$ converges. There's a Laplacian ∇^2 on a compact Lie group - coming from invariant metric & thus for $a > 0$ an operator

$$e^{-aH}: L^2(G) \longrightarrow L^2(G)$$

where $H = -\nabla^2$ has all nonnegative eigenvalues, one for each irrep of G :

$$L^2(G) = \bigoplus_{\rho \in \text{Irrep}(G)} \rho \otimes \rho^* \quad (\text{Peter-Weyl Thm})$$

Using this, we define for each $\alpha > 0$,

$$m_\alpha : L^2(G) \otimes L^2(G) \longrightarrow L^2(G)$$

$$\psi \otimes \varphi \longmapsto e^{-\alpha H}(\psi * \varphi)$$

This makes

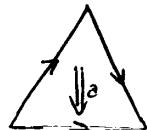
$$\text{tr}(L_\psi L_\varphi)$$

converge, where

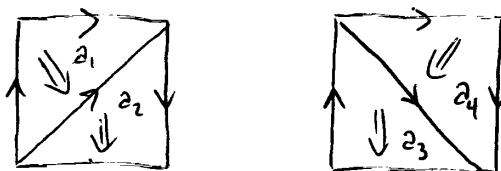
$$L_\psi : L^2(G) \longrightarrow L^2(G)$$

$$\varphi \longmapsto e^{-\alpha H} \psi * \varphi$$

We think of this m_α as related to a triangle:



where a is the area of the triangle. Luckily:

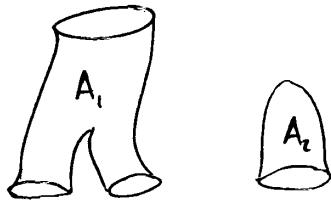


holds if the areas match

$$a_1 + a_2 = a_3 + a_4$$

So: we get not a topological quantum field theory, but an "area-logical field theory" - where we get well defined operators

from triangulated manifolds cobordisms equipped with total area
for each component:



So we get:

$$Z: 2\text{Cob}_{\text{with areas!}} \rightarrow \text{Vect}$$

Witten showed that this approach gives $Z(M) \in \mathbb{C}$ for a closed manifold with area. This agrees with the "symplectic structure on moduli space" approach in the limit as $\text{area} \rightarrow 0$, when the limit exists. (modulo some fudge factor involving Euler characteristic of M)