Quantum Gravity Seminar
Homework 1
Spring 2007

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C^∞ has pullbacks.

Definition of a smooth space

A smooth space is a set X equipped with, for each convex set C, a set of plots

$$\phi\colon C \to X$$

such that

- Given a plot $\phi\colon C \to X$ and a smooth map $f\colon C' \to C$ between convex sets, $\phi \circ f\colon C' \to X$ is a plot.
- Given inclusions $i_\alpha\colon C_\alpha \to C$ such that $\{C_\alpha\}$ is an open cover of C, given $\phi\colon C \to X$, then $\phi \circ i_\alpha\colon C_\alpha \to X$ are plots for every α implies $\phi\colon C \to X$ is a plot.
- Every map from a point (in $\mathbf{R}^n$) to X is a plot.

The pullback

Let X,Y,Z be smooth spaces with smooth maps

$$f\colon X \to Z \quad and \quad g\colon Y \to Z.$$

Now consider the product X×Y, a smooth space, where a function

$$\phi\colon C \to X \times Y$$

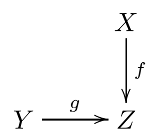
is a plot if and only if

$$C \xrightarrow{\phi} X \times Y \xrightarrow{p_1} X$$

$$C \xrightarrow{\phi} X \times Y \xrightarrow{p_2} Y$$

are plots.

We will show that the pullback of the diagram



is the subset of X×Y

$$P \;=\; \{z \in X \times Y \mid fp_1(z) \;=\; gp_2(z)\},$$

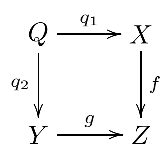
where a plot $\phi\colon C \to P$ is any function such that

$$C \xrightarrow{\phi} P \hookrightarrow X \times Y$$

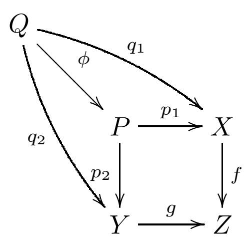
is a plot.

The universal property

Suppose Q, a smooth space, is a competitor for the title of 'the' pullback. Then it must come equipped with smooth maps q_1, q_2 such that



commutes. Consider the map $\phi\colon Q \to P$ defined by $r \longmapsto (q_1(r), q_2(r))$. By construction this is the unique map such that the following diagram commutes.



So we just need to show that ϕ is a morphism in our category, i.e. that it is smooth. Recalling the definition, ϕ is smooth if for every plot $\alpha\colon C \to Q$, $\phi \circ \alpha\colon C \to P$ is a plot. Since P is a subset of the product space $X \times Y$, $\phi \circ \alpha$ is a plot if and only if

$$C \xrightarrow{\alpha} Q \xrightarrow{\phi} P \xrightarrow{p_1} X$$

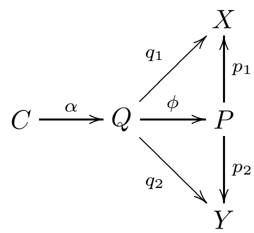
$$C \xrightarrow{\alpha} Q \xrightarrow{\phi} P \xrightarrow{p_2} Y$$

are plots. This follows from the fact that

$$C \xrightarrow{\alpha} Q \xrightarrow{q_1} X$$

$$C \xrightarrow{\alpha} Q \xrightarrow{q_2} Y$$

are plots and that



commutes.