

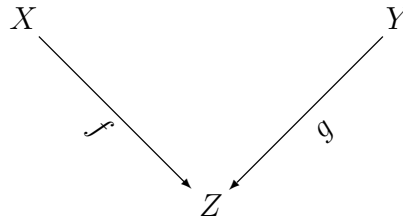
Dr. Baez

April 17, 2007

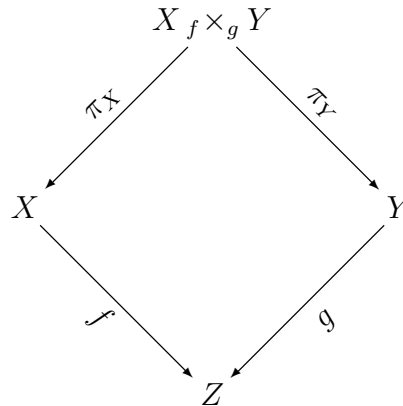
Christopher Walker

Exercise 1. Prove that the category C^∞ of “smooth spaces” and “smooth maps” has pullbacks.

Proof. Let the following be a diagram of smooth spaces.



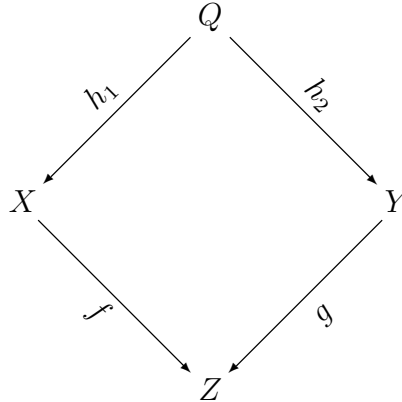
Define $X_f \times_g Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$. We have already seen that products exist in C^∞ and that subsets of smooth spaces are smooth spaces. Thus this smooth space exist in our category. Also define π_X and π_Y as the canonical projections onto X and Y , respectively. We claim now that the following diagram is a pullback of the diagram above.



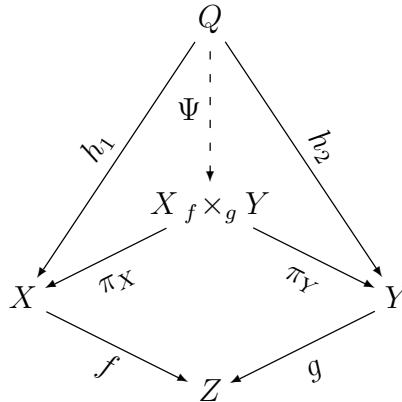
By the definition of $X_f \times_g Y$ we see that for all $(x, y) \in X_f \times_g Y$, we have:

$$\begin{aligned}
 f \circ \pi_X(x, y) &= f(x) \\
 &= g(y) \\
 &= g \circ \pi_Y(x, y)
 \end{aligned}$$

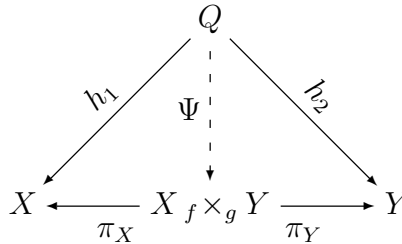
So the diagrams commutes. We then need to show that this object is “universal” for all such objects. This means that given another commutative diagram:



we need to show that there exist a unique morphism $\Psi : Q \rightarrow X_f \times_g Y$ making the following combined diagram commute.



The unique morphism we must have is $\Psi(q) = (h_1(q), h_2(q))$ for all $q \in Q$. First, $(h_1(q), h_2(q)) \in X_f \times_g Y$ since, by definition of h_1 and h_2 , $f \circ h_1 = g \circ h_2$. Ψ is a smooth map since for any plot $\phi : C \rightarrow Q$, $\Psi \circ \phi = (h_1 \circ \phi, h_2 \circ \phi)$ is a product of plots, and thus a plot. Most of this big diagram commutes from the previous diagrams. The only parts we need to check are the following triangles.



Fortunately, (since we really didn't have any choice) we defined Ψ to do exactly what we need, since $\pi_X \circ \Psi = h_1$ and $\pi_Y \circ \Psi = h_2$. So we have our pullback in C^∞ . \square