Math 260 - Quantization and Cohomology Seminar Spring 2007

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Exercise 1. *Prove that the category C^∞ of "smooth spaces" and "smooth maps" has pullbacks.*

Proof. Let the following be a diagram of smooth spaces.

$$\begin{tikzcd} X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

Define $X_f \times_g Y = \{(x, y) \in X \times Y \mid f(x) = g(y)\}$. We have already seen that products exist in C^∞ and that subsets of smooth spaces are smooth spaces. Thus this smooth space exist in our category. Also define π_X and π_Y as the canonical projections onto X and Y, respectively. We claim now that the following diagram is a pullback of the diagram above.

$$\begin{tikzcd} & X_f \times_g Y \arrow[dl, "\pi_X"'] \arrow[dr, "\pi_Y"] & \\ X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\ & Z & \end{tikzcd}$$

By the definition of $X_f \times_g Y$ we see that for all $(x, y) \in X_f \times_g Y$, we have:

$$\begin{aligned} f \circ \pi_X(x, y) &= f(x) \\ &= g(y) \\ &= g \circ \pi_Y(x, y) \end{aligned}$$

So the diagrams commutes. We then need to show that this object is "universal" for all such objects. This means that given another commutative diagram:

$$
\begin{tikzcd}
& Q \arrow[dl, "h_1"'] \arrow[dr, "h_2"] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}
$$

we need to show that there exist a unique morphism $\Psi : Q \to X\ {}_f\!\times_g Y$ making the following combined diagram commute.

$$
\begin{tikzcd}
& Q \arrow[ddl, "h_1"'] \arrow[ddr, "h_2"] \arrow[d, dashed, "\Psi"'] & \\
& X\ {}_f\!\times_g Y \arrow[dl, "\pi_X"] \arrow[dr, "\pi_Y"'] & \\
X \arrow[dr, "f"'] & & Y \arrow[dl, "g"] \\
& Z &
\end{tikzcd}
$$

The unique morphism we must have is $\Psi(q) = (h_1(q), h_2(q))$ for all $q \in Q$. First, $(h_1(q), h_2(q)) \in X\ {}_f\!\times_g Y$ since, by definition of h_1 and h_2, $f \circ h_1 = g \circ h_2$. Ψ is a smooth map since for any plot $\phi : C \to Q$, $\Psi \circ \phi = (h_1 \circ \phi, h_2 \circ \phi)$ is a product of plots, and thus a plot. Most of this big diagram commutes from the previous diagrams. The only parts we need to check are the following triangles.

$$
\begin{tikzcd}
& Q \arrow[dl, "h_1"'] \arrow[dr, "h_2"] \arrow[d, dashed, "\Psi"'] & \\
X & X\ {}_f\!\times_g Y \arrow[l, "\pi_X"] \arrow[r, "\pi_Y"'] & Y
\end{tikzcd}
$$

Fortunately, (since we really didn't have any choice) we defined Ψ to do exactly what we need, since $\pi_X \circ \Psi = h_1$ and $\pi_Y \circ \Psi = h_2$. So we have our pullback in C^∞. $\square$