Quantization and Cohomology

SPRING 2007

We've been starting with a category C equipped with an "action" functor:

$$S : C \longrightarrow (\mathbb{R}, +)$$

& doing classical and quantum mechanics using this. To do quantum mechanics, we formed

$$e^{iS} : C \longrightarrow (U(1), \cdot)$$

& did path integrals — integrals over objects and/or morphisms of C — so we needed something like a measure space (or perhaps a generalized measure space) of objects and a (generalized) measure space of morphisms. In classical mechanics, instead of integrating, we <u>minimize</u> the action, (Note: minimization requires <u>no</u> extra structure on C) or find critical points of the action (Note: this requires something like a "smooth structure" on the sets $\mathrm{Ob}(C)$ of objects and (especially) $Mor(C)$ of morphisms. For example, we might want $Ob(C)$, $Mor(C)$ or each $hom(x,y) \subseteq Mor(C)$ to be smooth manifolds. If $hom(x,y)$ were a manifold we could demand that $S: hom(x,y) \to \mathbb{R}$ be smooth, & look for $\gamma \in hom(x,y)$ with

$$dS(\gamma) = 0.$$

Alas, in examples $hom(x,y)$ is usually a more general "smooth space".

<u>Example</u>: The path groupoid PM of a manifold M has:

- points of M as objects: $Ob(PM) = M$
- thin homotopy classes of smooth paths $\gamma: [0,1] \to M$ which are constant near 0 and 1 as morphisms.

Here a <u>thin homotopy</u> between $\gamma_0: [0,1] \to M$ & $\gamma_1: [0,1] \to M$ is a smooth map:

$$H: [0,1] \times [0,1] \longrightarrow M$$

s.t.

$$\left.\begin{aligned} 1)\ & H(t,0) = \gamma_0(t) \\ 2)\ & H(t,1) = \gamma_1(t) \end{aligned}\right\} \text{(homotopy)}$$

3) $H(0,s)$ is indep. of s

4) $H(1,s)$ is indep. of s

AND 5) H is <u>thin</u>, i.e.

rank $dH \leq$, or equivalently

$\det(dH) = 0$, $\forall (t,s)$

Thin homotopies can do this:

Also, any reparameterization is a thin homotopy, so the obvious composition of morphisms in PM is associative & has identities & inverses — so PM is a groupoid.

If α is a 1-form on M we get a functor

$$S: PM \longrightarrow (\mathbb{R}, +)$$

sending every object of PM to the one object of $(\mathbb{R},+)$ & every morphism $[\gamma]$ to

$$S([\gamma]) = \int_\gamma \alpha \in \mathbb{R}$$

If γ_0 & γ_1 are thinly homotopic

$$\int_{\gamma_0} \alpha - \int_{\gamma_1} \alpha = \int_H d\alpha = 0$$

(since $\det(dH) = 0$

so S is well defined

This kind of example arises all over in classical mechanics, esp. when M is a cotangent bundle & α is the canonical 1-form.

We would like to say that PM is a "smooth category" & $S: PM \longrightarrow (\mathbb{R}, +)$ is a "smooth functor" & $\hom(x,y) \in \mathrm{Mor}(PM)$ is a "smooth space" so we can find critical points. Suppose we knew what "smooth spaces" and "smooth maps between smooth spaces" were. They'd better form a category, C^∞ say. Then, what's a "smooth category" and a "smooth functor"?

A smooth category C should have:

1) a smooth space of objects,
$\mathrm{Ob}(C) \in C^\infty$
2) a smooth space of morphisms
$\mathrm{Mor}(C) \in C^\infty$
3) ~~A~~ A smooth identity-assigning map $i: \mathrm{Ob}(C) \rightarrow \mathrm{Mor}(C)$,
i.e. i is a morphism in C^∞

4) Smooth source and target maps

$$s, t : \mathrm{Mor}(C) \longrightarrow \mathrm{Ob}(C)$$

5) The "composition" map is smooth:

$$\circ : \mathrm{Mor}(C)\,{}_t\!\times_s \mathrm{Mor}(C) \longrightarrow \mathrm{Mor}(C)$$

where $\mathrm{Mor}(C)\,{}_t\!\times_s \mathrm{Mor}(C)$ is the smooth space of composable pairs of morphisms in C, i.e. the pullback of this diagram

$$\begin{tikzcd} & \mathrm{Mor}(C) \arrow[d, "s"] \\ \mathrm{Mor}(C) \arrow[r, "t"'] & \mathrm{Ob}(C) \end{tikzcd}$$

i.e. the universal object making

$$\begin{tikzcd} \mathrm{Mor}(C)\,{}_s\!\times_t \mathrm{Mor}(C) \arrow[r] \arrow[d] & \mathrm{Mor}(C) \arrow[d, "s"] \\ \mathrm{Mor}(C) \arrow[r, "t"'] & \mathrm{Ob}(C) \end{tikzcd}$$

commute. (So C^∞ should have pullbacks or at least this pullback. The category of smooth manifolds and smooth maps doesn't have pullbacks!)

6) The associative laws, written as a commutative diagram in C^{∞}.

7) The left & right unit laws.

8) The source and target of a composite morphism are what they should be.