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Cohomology & Computation

SPRING 2007

Almost any algebraic gadget (ring, group, monoid, ...) can be described using a "presentation" — generators and relations — i.e. as some quotient of a "free" gadget. But there are also "relations between relations", or "syzygies." A very primitive example — generically if you have 3 linear equations in 3 unknowns you get a unique solution; but sometimes you get more solutions, e.g.

$$
\begin{aligned}
x + y + z &= 0 \\
2x + y \phantom{{}+z} &= 0 \\
3x + 2y + z &= 0
\end{aligned}
$$

Here we get a 1d space of solutions since we have a relation between relations:

$$
\begin{array}{r}
x + y + z = 0 \\
+ \quad 2x + y \phantom{{}+z} = 0 \\
\hline
3x + 2y + z = 0
\end{array}
$$

i.e. a linear dependence between equations

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Here we started with the free vector space $\langle x, y, z\rangle$ on x, y, z. Imposing the equations is taking a quotient by a subspace, which if we have 3 equations is the image of some linear map

$$\langle a,b,c\rangle \xrightarrow{\quad d_1 \quad} \langle x, y, z\rangle$$

$$\begin{aligned} a &\longmapsto x + y + z \\ b &\longmapsto 2x + y \\ c &\longmapsto 3x + 2y + z \end{aligned}$$

The space of solutions of our equations is (isomorphic to)

$$S := \frac{\langle x,y,z\rangle}{\operatorname{im} d_1}$$

Then

relations — generators — gadget being presented

$$\langle a,b,c\rangle \xrightarrow{d_1} \langle x,y,z\rangle \xrightarrow{\quad d_0 \quad} S \xrightarrow{d_{-1}} 0$$

quotient map

is exact :

$\operatorname{im} d_1$ is exactly $\ker d_0$

$\operatorname{im} d_0$ is exactly $\ker d_{-1}$ (d_0 is onto)

If d_1 were injective — which it's not in our example since we have relations between relations — then we get a longer exact sequence:

$$0 \xrightarrow{d_2} \langle a,b,c\rangle \xrightarrow{d_1} \langle x,y,z\rangle \xrightarrow{d} S \longrightarrow 0$$

where $\mathrm{im}\, d_2 = \ker d_1$ says d_1 is injective. In this case, we'd have

$$\dim\langle a,b,c\rangle - \dim\langle x,y,z\rangle + \dim S = 0$$

so $\dim S = 0$. But in <u>our</u> example, something more interesting happens: a <u>syzygy</u> or relation between relations. We have

"equation a + equation b = equation c"

so we get a longer exact sequence

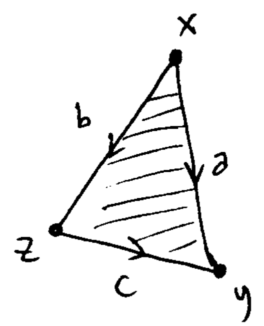
$$0 \to \langle u\rangle \longrightarrow \langle a,b,c\rangle \longrightarrow \langle x,y,z\rangle \longrightarrow S \longrightarrow 0$$

where $d_2 u = a+b-c$. The fact that $\mathrm{im}\, d_2 = \ker d_1$ says d_2 exactly captures the relations between relations.

We now get

$$\dim\langle u\rangle - \dim\langle a,b,c\rangle + \dim\langle x,y,z\rangle - \dim S = 0$$

so $\dim S = 1$. This alternating sum is an Euler characteristic:



$$x \underset{a}{=} y \qquad x \underset{b}{=} z \qquad y \underset{c}{=} z$$

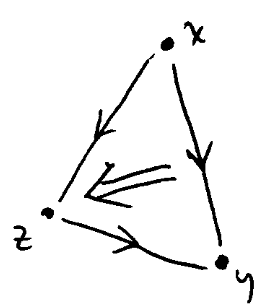
$$a = b + c$$

This big island has Euler characteristic

$$3 - 3 + 1 = 1$$

0-dim islands	1-dim bridges	2-dim bridges between bridges
generators	relations	syzygies
objects	morphisms	2-morphisms

In 2-category theory we draw diagrams like



In fact almost every algebraic gadget has a "canonical presentation" with

all possible generators
all possible relations.

In this case, there will generally be lots of syzygies, etc. So we might as well include

all possible syzygies
all possible ... um, "syzyzygies"
⋮
all possible "sy(zy)^{n}gies"
⋮

In fact we can learn a lot about our gadget by building a topological space that encodes all generators, relations, relns between relns, etc. We draw a dot for each elt of the gadget freely generated by our generators, an edge for each reln, a triangle for each syzygy, etc.

The co/homology of this space contains a lot of important information about the gadget. If you look at Mac Lane's <u>Homology</u> or H. Cartan & Eilenberg's <u>Homological Algebra</u>, you'll see lots of information about co/homology of

groups
rings
Lie algebras
⋮

We've begun to see this kind of space already (last quarter) when studying presentations of monoids:

$$\langle x, y, z \mid xyz \overset{a}{\Longrightarrow} xy \ ,\ yzx \overset{b}{\Longrightarrow} yx \rangle$$

Here we drew "dots" & "edges":

$$\begin{tikzcd} xyzx \arrow[d, Rightarrow, bend right, "a\cdot x"'] \arrow[d, Rightarrow, bend left, "x\cdot b"] \\ xyx \end{tikzcd}$$

but we didn't go on to consider 2-simplices corresponding to "relations between proofs".

But now, using ideas from cohomology, we can go on forever & build a big infinite dimensional space.