10 Apr 2007

Smooth Spaces & Smooth Categories

Last time we posed a question: what's a "smooth functor"

$$S : C \longrightarrow \mathbb{R} \quad ?$$

For this we need C to be a "smooth category", which we succeeded in defining given a category C^∞ of "smooth spaces" & "smooth maps." More generally, given a category K we can define a category (say C) in K to consist of

- $Ob(C) \in K$
- $Mor(C) \in K$
- $s, t : Mor(C) \longrightarrow Ob(C)$ (i.e. $s, t \in \hom_K(Mor(C), Ob(C))$)
- $i : Ob(C) \longrightarrow Mor(C)$
- $\circ : Mor(C)\, {}_t\!\times_s Mor(C) \longrightarrow Mor(C)$

satisfying the usual category theory axioms.

This trick — taking a definition & replacing sets and functions by objects and morphisms of K — is called internalization (in K). The result is often called an internal category (in K) or simply a category in K.

This works best if K has pullbacks, so we can write "$\mathrm{Mor}(C) {}_s\!\times_t \mathrm{Mor}(C)$" and know it's defined.
The category $K = \mathrm{Set}$ has pullbacks:

$$\begin{tikzcd}
X {}_f\!\times_g Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d, "f"] \\
Y \arrow[r, "g"'] & Z
\end{tikzcd}$$

where in this case

$$X {}_f\!\times_g Y = \{(x,y) \in X \times Y \ : \ f(x) = g(y)\}$$

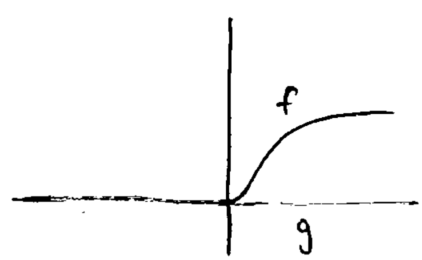
The category Diff of smooth manifolds & smooth maps doesn't have pullbacks. For a rough idea why, consider

$$\begin{tikzcd}
\{(x,y) : f(x) = g(y)\} \arrow[r] \arrow[d] & \mathbb{R} \arrow[d, "f"] \\
\mathbb{R} \arrow[r, "g"'] & \mathbb{R}
\end{tikzcd}$$

where

$$f(x) = \begin{cases} e^{-1/x} & x > 0 \\ 0 & x \leq 0 \end{cases}$$

$$g(x) = 0$$

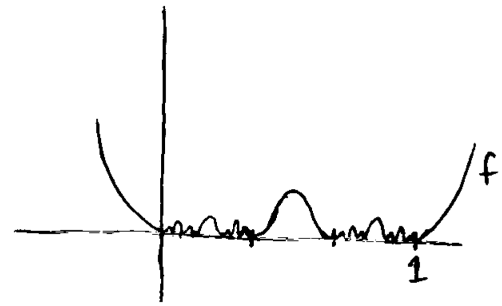


Here $\{(x,y) : f(x) = g(y)\} = \{(x,y) \text{ s.t. } x \leq 0\}$ is a manifold with boundary — <u>not</u> a smooth manifold.

A more sophisticated example

$f(x) = 0$ only on the Cantor set

$g(x) = 0$



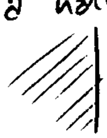
So, let's introduce a bigger category of smooth spaces that does have pullbacks, following Grothendieck's dictum:

> "A nice category with some bad objects is better than a bad category with only nice objects."

He used this to invent "schemes", generalizing "algebraic varieties," revolutionizing algebraic geometry. Let's do the same for differential geometry.

So, what's a smooth space? Following Chen's ideas:

Def: A <u>convex set</u> is a convex subset of $\mathbb{R}^n$

e.g. $\mathbb{R}^n$ , a half-space , a quarter-space , etc.

Def: A function $f: C \to C'$ between convex sets is <u>smooth</u> if it has continuous nth derivatives $\forall n \geq 0$, defined in the usual way.

Def: A smooth space is a set X equipped with, for each convex set C, a set of plots

$$\varphi : C \to X$$

(which we think of as 'smooth'), s.t.

1) Given a plot $\varphi : C \to X$ & a smooth map $f : C' \to C$ between convex sets, $\varphi \circ f : C' \to X$ is a plot.

2) Given inclusions $i_\alpha : C_\alpha \to C$ such that $\{C_\alpha\}$ is an open cover of C, given $\varphi : C \to X$, then $\varphi \circ i_\alpha : C_\alpha \to X$ are plots $\forall \alpha$ $\Longrightarrow$ $\varphi : C \to X$ is a plot.

3) Every map from a point (in $\mathbb{R}^n$) to X is a plot.

Def: Given smooth spaces X & Y a function $f: X \to Y$ is a smooth map if for every plot $\varphi : C \to X$, $f \circ \varphi : C \to Y$ is a plot in Y.

Examples:

1) Any convex set C becomes a smooth space where the plots $\varphi : C' \longrightarrow C$ are just the smooth maps (as defined earlier).

2) Any set X has a discrete smooth structure such that the plots $\varphi : C \rightarrow X$ are just the constant functions

3) Any set X has an indiscrete smooth structure where every function $\varphi : C \longrightarrow X$ is a plot.

4) Any smooth manifold X becomes a smooth space where $\varphi : C \rightarrow X$ is a plot iff φ is smooth in the usual sense of smooth manifolds. If X, Y are manifolds, $f : X \rightarrow Y$ is smooth according to our new definition iff it's smooth in the usual sense.

5) The product $X \times Y$ of smooth spaces becomes a smooth space where a function $\varphi : C \longrightarrow X \times Y$ is a plot iff $C \xrightarrow{\varphi} X \times Y \xrightarrow{p_1} X$
& $C \xrightarrow{\varphi} X \times Y \xrightarrow{p_2} Y$ are plots.

In fact our category C^∞ of smooth spaces & smooth maps has products, given as above.

6) The disjoint union $X+Y$ of smooth spaces becomes a smooth space where a function $\varphi : C \to X+Y$ is a plot iff either

$$\begin{tikzcd} & X \arrow[d, "i_1"] \\ C \arrow[ur, "\varphi_1"] \arrow[r, "\varphi"'] & X+Y \end{tikzcd}$$

commutes for some plot $\varphi_1 : C \to X$ or

$$\begin{tikzcd} & Y \arrow[d, "i_2"] \\ C \arrow[ur, "\varphi_2"] \arrow[r, "\varphi"'] & X+Y \end{tikzcd}$$

commutes for some plot $\varphi_2 : C \to Y$.

In fact, C^∞ has coproducts.

7) Any subset $Y \subseteq X$ of a smooth space X becomes a smooth space, where a plot $\varphi : C \to Y$ is any function s.t.

$$C \xrightarrow{\varphi} Y \hookrightarrow X$$

is a plot in X. So, e.g. the Cantor set is a smooth space.

<u>HW</u> — Show that C^∞ has pullbacks.