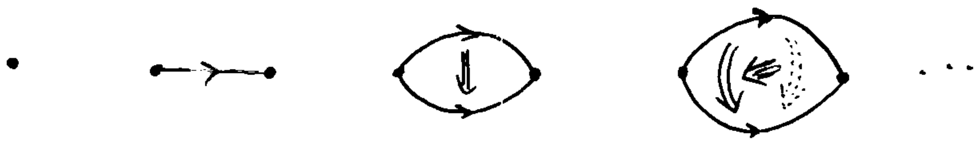
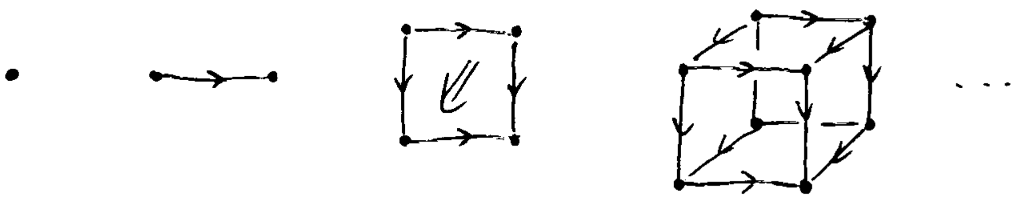
Cohomology & the Category of Simplices

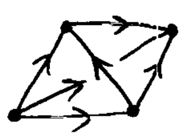
Why are simplices better (in many ways) than globes:



or cubes:

 ?

One answer is that a simplex



is a special sort of category — namely, a finite totally ordered set! The set is the set of vertices; the $<$ relation gives the edges. Every finite totally ordered set is isomorphic to a finite <u>ordinal</u>:

$$\begin{aligned} 0 &= \{\ \} \\ 1 &= \{0\} \\ 2 &= \{0,1\} \\ 3 &= \{0,1,2\} \\ &\vdots \end{aligned}$$

where $<$ is now $\in$.
(or $\subseteq$)

There's a category Δ_{alg} – the algebraist's category of simplices – where the objects are finite totally ordered sets (or simplices) & the morphisms are order preserving functions $f: S \to T$, i.e. functions with

$$x \leq y \implies f(x) \leq f(y)$$

(If we think of simplices as (toset) categories, order-preserving functions are just functors.) A typical morphism in Δ_{alg} looks like:

$$\begin{tikzcd}[column sep=small]
\{ & 0 \arrow[dr] & 1 \arrow[d] & 2 \arrow[dl] & 3 \arrow[dr] & 4 \arrow[d] & 5 \arrow[dr] & \} & & = 6 \\
\{ & 0 & 1 & 2 & 3 & 4 & 5 & 6 & \} & = 7
\end{tikzcd}$$

Δ_{alg} is generated by the objects $0, 1, 2, \ldots$ and certain special morphisms like:

$$\begin{tikzcd}[column sep=small]
 & 0 \arrow[d] & 1 \arrow[d] & 2 \arrow[dr] & 3 \arrow[dr] & 4 \arrow[dr] & \\
d_2 & 0 & 1 & 2 & 3 & 4 & 5
\end{tikzcd}$$

1-1 and almost onto

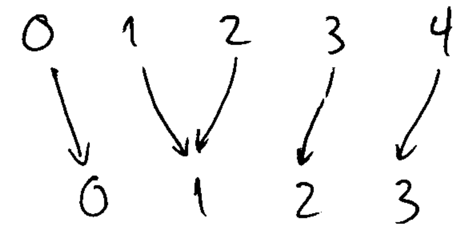
Or more generally we need

$$d_j : n \longrightarrow n+1 \qquad 0 \leq j \leq n$$

– the order preserving map whose image only fails to contain j.

We also need ones like:

e_1

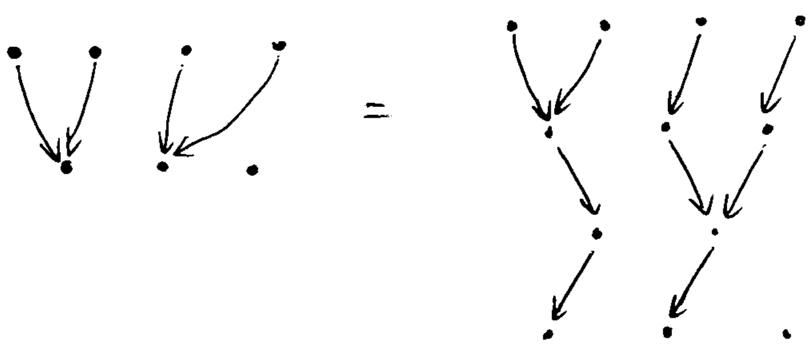


onto and almost 1-1

or more generally

$$e_j : n+1 \longrightarrow n \qquad 0 \le j \le n-1$$

— the order preserving map for which $j \in n$ is the only element with two preimages. By "generated" we mean any order preserving function $f : n \to m$ is a composite of maps d_j, e_j. E.g.

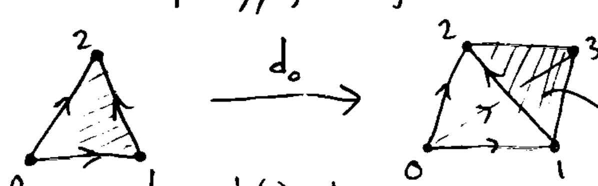


e_0

e_1

d_2

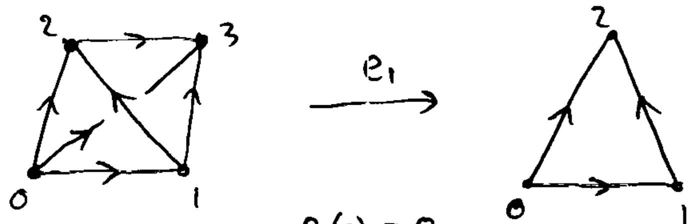
In terms of topology, $d_j : n \longrightarrow n+1$ is a <u>face</u> map:



$d_0(0) = 1$
$d_0(1) = 2$
$d_0(2) = 3$

face that lacks the 0th vertex.

Similarly $e_j : n+1 \to n$ is a degeneracy map

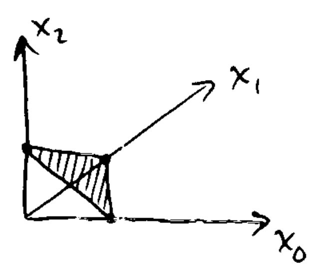


$e_1(0) = 0$
$e_1(1) = 1$
$e_1(2) = 1$
$e_1(3) = 2$

e_j squashes vertices j & $j+1$ down to the vertex j.

We can turn n into a space, the <u>standard</u> $(n-1)$-simplex:

$$\Delta_{n-1} = \left\{ (x_0, \dots, x_{n-1}) \in \mathbb{R}^n \;:\; x_i \geq 0 \;\&\; \sum x_i = 0 \right\}$$



Then any morphism $f : n \to m$ can be turned into an affine map

$$\Delta_{f-1} : \Delta_{n-1} \longrightarrow \Delta_{m-1}$$

a silly name (but a systematic one, since Δ_{-1} is a functor!)

— the unique affine map sending the jth vertex of Δ_{n-1} to the $f(j)$-th vertex of Δ_{m-1}.

This process of turning ordinals into spaces & order-preserving maps into continuous maps is a functor:

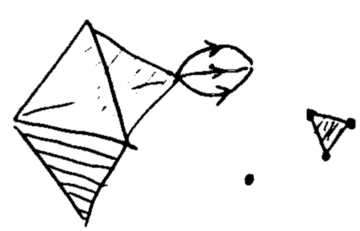
$$\Delta_{\_-1} : \Delta_{alg} \longrightarrow Top$$

When $n=0$, this functor gives "the standard -1-simplex" which is $\emptyset \subseteq \mathbb{R}^0$. If this freaks us out, we restrict attention to Δ_{top}, the category of <u>nonempty</u> totally ordered sets & order-preserving functions. This is the topologist's category of simplices. Topologists call this Δ and call Δ_{alg} the <u>augmented</u> category of simplices.

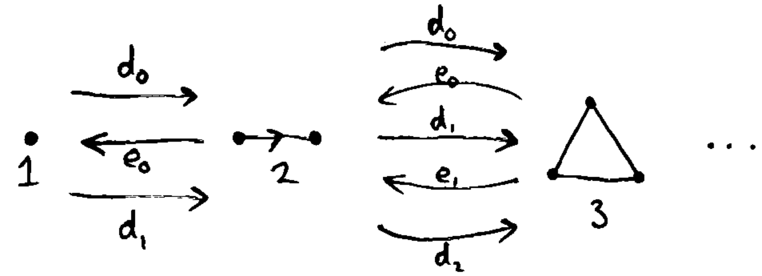
Def: A simplicial set is a functor

$$F : \Delta_{top}^{op} \longrightarrow Set.$$

Claim: such a thing looks sort of like:



Δ_{top} looks sort of like:



Δ_{top}^{op} is just the same diagram with all the arrows reversed.

So a simplicial set F looks like

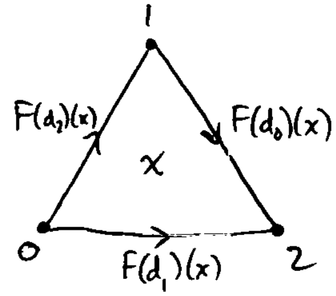
$$
\begin{tikzcd}
F(1) \arrow[r, "F(e_0)"] & F(2) \arrow[l, bend right, "F(d_0)"'] \arrow[l, bend left, "F(d_1)"] \arrow[r, bend left=20, "F(e_0)"] \arrow[r, bend right=20, "F(e_1)"'] & F(3) \arrow[l, bend right=50, "F(d_0)"'] \arrow[l, "F(d_1)"'] \arrow[l, bend left=50, "F(d_2)"] & \cdots
\end{tikzcd}
$$

set of vertices — set of edges — set of triangles

Given any edge $\bullet \xrightarrow{x} \bullet$ we get two vertices:

$$F(d_1)(x) \bullet \xrightarrow{x} \bullet F(d_0)(x)$$

Similarly:



Usually people write

$$\partial_j = F(d_j)$$
$$\varepsilon_j = F(e_j)$$

So a simplicial set is like

$$\begin{tikzcd}
F(1) \arrow[r, "\varepsilon_0"] & F(2) \arrow[l, bend right=50, "\partial_0"'] \arrow[l, bend left=50, "\partial_1"] \arrow[r, bend left=20, "\varepsilon_0"] \arrow[r, bend right=20, "\varepsilon_1"'] & F(3) \arrow[l, "\partial_1"'] \arrow[l, bend right=60, "\partial_0"'] \arrow[l, bend left=60, "\partial_2"] & \cdots
\end{tikzcd}$$

The way cohomology of spaces works:

$$\mathrm{Top} \xrightarrow[\text{singular nerve}]{S} \hom(\Delta^{op}, \mathrm{Set}) \xrightarrow[\text{free abelian group}]{F \circ -} \underset{\text{simplicial abelian gp.}}{\hom(\Delta^{op}, \mathrm{AbGp})} \underset{\text{Dold-Kan Theorem}}{\simeq} [\text{Chain complexes}]$$