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# The Path Groupoid of a Smooth Space

We've defined a notion of "smooth space" — a set $X$ equipped with a collection of plots $\phi: C \to X$ for convex sets $C$, satisfying some properties. We've also defined a notion of "smooth map" — a function $f: X \to Y$ s.t.

$$\phi: C \longrightarrow X \text{ is a plot}$$
$$\Downarrow$$
$$f \circ \phi: C \longrightarrow Y \text{ is a plot.}$$

We claimed that there's a category $C^\infty$ of smooth spaces & smooth maps. We listed a bunch of nice properties of $C^\infty$; here are some more:

(dual to #7)

8) Any quotient $Y = X/\sim$, where $X$ is a smooth space and $\sim$ is an equivalence relation on $X$, becomes a smooth space where a plot in $Y$ is a composite

$$\begin{tikzcd} C \arrow[r, "\phi"] \arrow[rr, bend left] & X \arrow[r, "p"] & Y = X/\sim \end{tikzcd}$$

where $\phi$ is any plot in $X$. This definition guarantees that the quotient map $p: X \to Y$ is smooth.

This definition also lets us see that $C^\infty$ has pushouts:

$$\begin{tikzcd} Z \arrow[r, "f"] \arrow[d, "g"'] & X \arrow[d, "i_1"] \\ Y \arrow[r, "i_2"'] & \dfrac{X+Y}{i_1(f(z)) = i_2(g(z))} \end{tikzcd}$$

The pushout is a quotient of a disjoint union (= coproduct).

9) $C^\infty$ is Cartesian closed: that is, given smooth spaces $X$ & $Y$, the set

$$C^\infty(X,Y) = \{ f : X \to Y : f \text{ is smooth} \}$$

can be made into a smooth space such that there's a natural isomorphism

$$C^\infty(X \times Y, Z) \cong C^\infty(X, C^\infty(Y,Z))$$
$$f \longmapsto \tilde{f}$$

where in this case

$$\tilde{f}(x)(y) = f(x,y).$$

To do this, we define a plot in $C^\infty(Y,Z)$, say

$$\phi : C \to C^\infty(Y,Z)$$

to be any map of the form $\tilde{f}$ where

$$f : C \times Y \to Z \qquad \text{(using currying in Set)}$$

is smooth.

One must check that $C^\infty(Y,Z)$ really is a smooth space and that then we get a 1-1 & onto map

$$C^\infty(X\times Y, Z) \longrightarrow C^\infty(X, C^\infty(Y,Z))$$

which turns out to be smooth.

Using properties 1-9 we can make a smooth category $PX$, the <u>path groupoid</u> of any smooth space $X$. This has:

- a smooth space of objects $Ob(PX) = X$
- a smooth space of morphisms $Mor(PX) = \frac{C^\infty([0,1],X)}{\sim}$

where $[0,1]$ is smooth since it's a convex set, $C^\infty([0,1],X)$ is a smooth space by Cartesian closedness and modding out by "thin homotopy" gives a smooth space $C^\infty([0,1],X)/\sim$. We also need to check that the source, target, identity-assigning, & composition maps are smooth.

Next we want to understand smooth functors

$$S : PX \longrightarrow (\mathbb{R},+)$$

since such a functor describes the "action" for some classical system.

Def: Given smooth categories $C$ & $D$, a smooth functor is:

- a smooth map $F: Ob(C) \longrightarrow Ob(D)$
- a smooth map $F: Mor(C) \longrightarrow Mor(D)$

such that usual properties of a functor hold, written as commutative diagrams:

1) $F$ preserves source for morphisms:

$$\begin{tikzcd} Mor(C) \arrow[r, "F"] \arrow[d, "s_C"'] & Mor(D) \arrow[d, "s_D"] \\ Ob(C) \arrow[r, "F"] & Ob(D) \end{tikzcd}$$

commutes.

similarly

2) $F$ preserves targets

3) $F$ preserves identity-assigning map

4) $F$ preserves composition.

(Just as you can define "categories in $K$" for any category $K$ with pullbacks, you can define "functors in $K$.")

Theorem: For any smooth space $M$, there's a 1-1 correspondence between smooth functors

$$S: PM \longrightarrow (\mathbb{R}, +)$$

and 1-forms $A$ on $M$ (defined below) given by

$$S([\gamma]) = \int_\gamma A .$$

**Def:** A $p$-form $\alpha$ on a smooth space $X$ assigns to each plot $\phi: C \longrightarrow X$ a $p$-form $\alpha_\phi$ on $C$ such that given a smooth map $f: C' \longrightarrow C$ between convex sets

$$f^* \alpha_\phi = \alpha_{\phi \circ f}$$

where $f^*$ is the operation of pulling back $p$-forms from $C$ to $C'$ along $f: C' \longrightarrow C$.

For the proof of the theorem see
"Higher Gauge Theory II" by Baez & Schreiber.