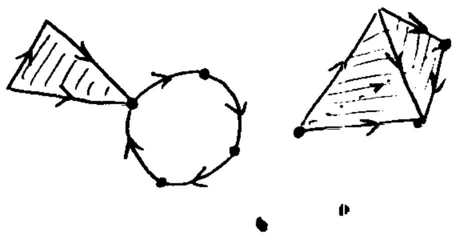
19 April 2007

Simplicial Sets & Cohomology

A simplicial set is something like:



It can have "holes" of different dimensions, and these are detected by co/homology:

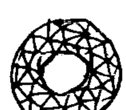
(triangulated 0-sphere)

0-dimensional holes
"0th cohomology" H^0

triangulated circle

1-dimensional holes
"1st cohomology" H^1

triangulated sphere



triangulated torus

2-dimensional holes
"2nd cohomology" H^2

We get simplicial sets from various sources, esp.:

1) Topology. You can take a sufficiently nice topological space and triangulate it to get a simplical set;
You can take any topological space X and get a simplicial set SX of all possible ways to map simplices into X:

2) Algebra. Given any presentation of (almost any) algebraic gadget — a group, a ring, a Lie algebra, etc — there's a way to build a simplicial set. E.g. given a group presentation

$$\langle x, y \mid xy = yx \rangle$$

(a presentation of $\mathbb{Z}^2$) we get a simplicial set with

- all elts of the free group on the generators as 0-simplices:

$$x^{-1}y \qquad x \qquad xyx$$

$$xy \qquad yx \qquad \text{etc.}$$

- 1-simplices coming from relations

$$xy \text{ ——— } yx$$

$$yxx \text{ ——— } xyx \qquad \text{etc.}$$

- 2-simplices coming from relations between relations (syzygies)

⋮

etc

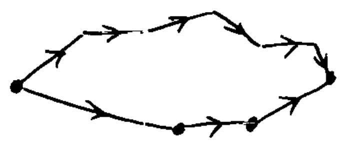
The holes in this simplicial set convey information about our gadget! Edge paths correspond to "calculations"

For example:

$$yxx \longrightarrow xyx \longrightarrow xxy$$

represents the process of proving $yxx = xxy$

1-dimensional holes arise when you have two nonhomotopic calculations with the same starting point and same endpoint:



Back to business:

We have a category

$$\Delta_{alg} = [\text{finite totally ordered sets}]$$

$$\begin{tikzcd}
0 \arrow[r, "d_0"] & 1 \arrow[r, bend left, "d_0"] \arrow[r, bend right, "d_1"'] & 2 \arrow[l, "i_0"'] \arrow[r, bend left=60, "d_0"] \arrow[r, "d_1"] \arrow[r, bend right=60, "d_2"'] & 3 \arrow[l, bend right, "i_0"'] \arrow[l, bend left, "i_1"] \arrow[r, bend left=60] \arrow[r, bend left=20] \arrow[r, bend right=20] \arrow[r, bend right=60] & 4 \arrow[l, bend right=40] \arrow[l] \arrow[l, bend left=40] & \cdots
\end{tikzcd}$$

& we have a functor

$$\begin{aligned} \Delta_{-1} : \Delta_{alg} &\longrightarrow \text{Top} \\ n &\longmapsto \Delta_{n-1} \end{aligned}$$

sending the ordinal n to the simplex with n vertices, aka. the $(n-1)$-simplex Δ_{n-1}.

Topologists find the (-1)-simplex Δ_{-1} terrifying, so they create

$$\Delta_{top} = [\text{finite totally ordered nonempty sets}]$$

$$\begin{tikzcd} 1 \arrow[r, shift left=3] \arrow[r, shift right=3] & 2 \arrow[l] \arrow[r, shift left=4] \arrow[r] \arrow[r, shift right=4] & 3 \arrow[l, shift left=2] \arrow[l, shift right=2] & \cdots \end{tikzcd}$$

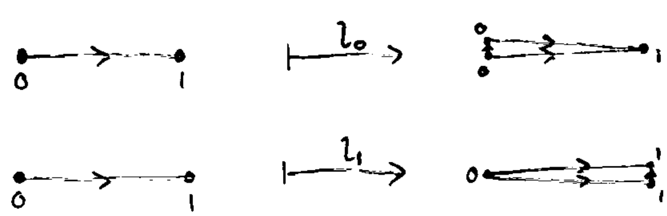
For them, a simplicial set is a functor

$$G : \Delta_{top}^{op} \longrightarrow Set$$

(& they call a functor $G : \Delta_{top}^{op} \to Set$ an augmented simplicial set) This looks like

$$\begin{tikzcd} G(1) \arrow[r, "\iota_0"] & G(2) \arrow[l, bend right, "\partial_0"'] \arrow[l, bend left, "\partial_1"] & \cdots \end{tikzcd}$$

where $G(n)$ is the set of $(n-1)$-simplices, & the functions ∂_i take an n-simplex & produce the $(n-1)$-simplex obtained by omitting the ith vertex, while ι_i are "degeneracies":



Given a space $X \in Top$, we can build a simplicial set SX — the singular simplicial set of X — as we described:

$$SX(n) = \{ \text{all maps } f : \Delta_{n-1} \to X \}$$
$$= \hom(\Delta_{n-1}, X)$$

(hom in the category Top.)

These $SX(n)$ are related by functions ∂_i, ι_i — the face and degeneracy maps. But really a simplicial set is a functor $G : \Delta_{top}^{op} \to Set$ and SX is just

$$\begin{tikzcd} \Delta_{top}^{op} \arrow[rr, "SX"] \arrow[dr, "\Delta_{-1}"'] & & Set \\ & Top^{op} \arrow[ur, "\hom(-,X)"'] & \end{tikzcd}$$

Similarly for any map $f : X \to Y$ between spaces we get a map between simplicial sets $Sf : SX \to SY$. We get a functor

$$S : Top \longrightarrow SimpSet$$

where SimpSet is the category of simplicial sets: functors $G : \Delta_{top}^{op} \to Set$ as objects & natural transformations

$$\begin{tikzcd} \Delta_{top}^{op} \arrow[r, bend left=50, "G"{name=U}] \arrow[r, bend right=50, "H"'{name=D}] & Set \arrow[from=U, to=D, Rightarrow, "\alpha"] \end{tikzcd}$$

as morphisms.

So:

$$\mathrm{SimpSet} = \hom(\Delta_{top}^{op}, \mathrm{Set})$$

In fact:

$$\mathrm{Top} \longrightarrow \hom(\Delta_{top}^{op}, \mathrm{Set})$$

comes from

$$\mathrm{Top} \times \Delta_{top}^{op} \longrightarrow \mathrm{Set}$$

$$(X, n) \longmapsto SX(n) = \hom(\Delta_{n-1}, X)$$

or better:

$$\Delta_{top}^{op} \times \mathrm{Top} \xrightarrow{\hom(\Delta_{-1}, =)} \mathrm{Set}.$$