Smooth Functors And Beyond

We've seen that if M is a smooth space (e.g. a manifold) then there is a smooth category $\mathcal{P}M$ where:

- objects are points $x \in M$
- morphisms $\gamma: x \to y$ are thin homotopy classes of smooth maps $f: [0,1] \to M$ with $f(0) = x$, $f(1) = y$ & f constant near 0 & 1.

We think of $\mathcal{P}M$ as a category of "configurations" & "processes" for some physical system. So to formulate the Lagrangian approach to the physics of this system, we need a smooth functor

$$S : PM \to \mathbb{R}$$

describing the "action" of any process.

Last time we saw:

Theorem - There's a 1-1 correspondence between smooth functors

$$S : PM \to \mathbb{R}$$

& 1-forms A on M, given by:

$$S(\gamma) = \int_\gamma A$$

where we pick a representative (path) for γ to define the integral.

Alas, this isn't general enough ... as we'll soon see.

To do quantum physics, what matters is not

$$S : PM \to \mathbb{R}$$

but the phase

$$e^{iS} : PM \longrightarrow U(1)$$

which has less information since

$$\exp : \mathbb{R} \longrightarrow U(1)$$

is many-to-one. In fact, e^{iS} is also sufficient to do classical physics!

$$\frac{d}{ds} S(\gamma_s) = 0$$

If we seek <u>critical points</u> of the action (instead of minima), we can work with e^{iS} instead of S:

$$\frac{d}{ds} e^{iS(\gamma_s)} = 0$$

(for all smooth homotopies γ_s of γ holding endpoints fixed).

The critical points of e^{iS} are the same as those of S, so this doesn't seem like a big deal.

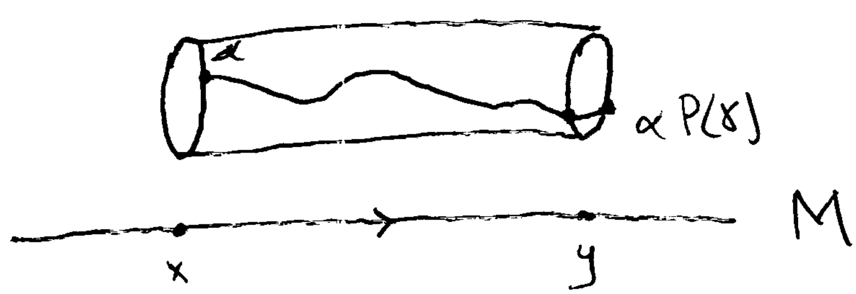
Theorem — There's a 1-1 correspondence between smooth functors

$$P : PM \to U(1)$$

& 1-forms A on M, given by:

$$P(\gamma) = e^{i\int_\gamma A}$$

Here P stands for "phase". So far the picture looks like:



For each point $x \in M$ we have a circle of possible phases for the system in configuration x, so we have a "trivial principal $U(1)$ bundle":

$$\begin{array}{ccc} M \times U(1) & \ni & (x, \alpha) \\ \big\downarrow \pi_1 & & \big\downarrow \pi_1 \\ M & & x \end{array}$$

Sitting over $x \in M$ we have a <u>fiber</u>

$$\pi_1^{-1}(x) \subseteq M \times U(1)$$

which is a circle – the set of possible phases our system could have at x.

This example is called "trivial" because each fiber "<u>is</u>" $U(1)$ – or is canonically isomorphic to $U(1)$:

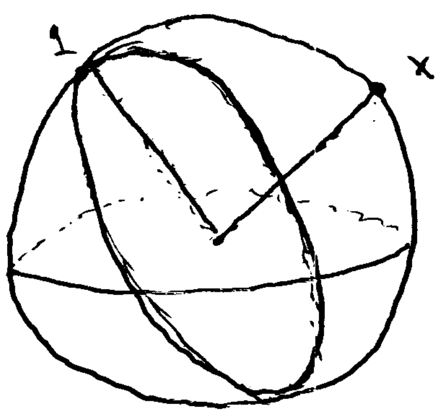
$$\pi_1^{-1}(x) = \{(x,\alpha) : \alpha \in U(1)\}$$

$$\Big\downarrow \wr$$

$$U(1)$$

where the isomorphism sends (x,α) to α.

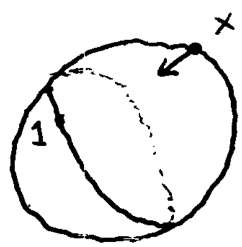
More interesting are the <u>nontrivial</u> principal $U(1)$ bundles:



$$M = S^2 = \mathbb{C}P^1$$

For example, let the fiber over x be the set of points in S^2 that are $\perp$ to x.

We can't smoothly identify all the fibers with $U(1)$ since that would produce a nowhere vanishing smooth vector field on S^2:



Now let's get a bit more formal.

What's the difference between a circle & the circle? The circle is $U(1) \subseteq \mathbb{C}$. A circle is a "$U(1)$-torsor". - a copy of $U(1)$ that's forgotten what the element 1 is.

Def. - For any group G, a **G-torsor** is a set X equipped with an action (a right action) of G:

$$\alpha : X \times G \longrightarrow X$$
$$(x, g) \longmapsto xg$$

s.t.

$$x1 = x$$

$$(xg)h = x(gh)$$

such that X is isomorphic to G as a space with right G-action: there's a bijection

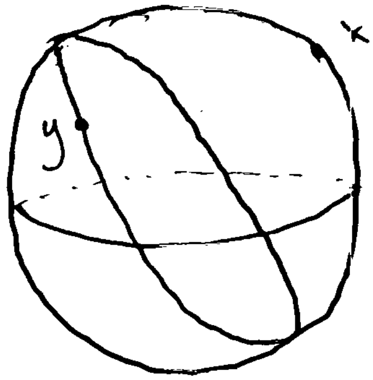
$$\beta : X \to G$$

s.t.

$$\beta(xg) = \beta(x)g \, .$$

If $G = U(1)$, the difference between right & left actions is inessential since $U(1)$ is abelian. More importantly, any circle equipped with the ability to rotate it by any phase $g \in U(1)$ is a $U(1)$-torsor.

E.g.



$$\begin{array}{ccc} P & & (x,y) \\ \pi \big\downarrow & & \big\downarrow \\ M = S^2 & & x \end{array}$$

if a point in P is a point $x \in S^2$ together with a point in the circle $\perp$ to x, & $\pi : P \to M$ is the obvious map, then $\pi^{-1}(x)$ is a $U(1)$-torsor.

* More precisely, $\pi^{-1}(x)$ <u>becomes</u> a $U(1)$-torsor after we pick a "right or left-hand rule" for rotating $y \in \pi^{-1}(x)$ by a phase $g \in U(1)$.

A principal $U(1)$ bundle is (among other things) a smooth space P with a smooth map

$$\begin{array}{c} P \\ \pi \big\downarrow \\ M \end{array}$$

such that each fiber $\pi^{-1}(x)$ $(x \in M)$ is equipped with the structure of being a $U(1)$-torsor.

Next time we'll <u>really</u> define "principal $U(1)$-bundle" & include a clause saying that the $U(1)$-torsor structure on $\pi^{-1}(x)$ varies smoothly with x.