Chain Complexes

We can get simplicial sets from topological spaces, algebraic gadgets, etc. To count the holes (of various dimensions) of a simplicial set, we process it:

$$\begin{tikzcd}
\left[\text{simplicial sets}\right] \arrow[d, "F"', "\substack{\text{compose with free abelian gp functor } \mathrm{Set}\to\mathrm{AbGp} \\ \text{(i.e. take formal } \mathbb{Z}\text{-linear combinations)}}"] \\
\left[\text{simplicial abelian groups}\right] \arrow[d, "Ch"', "\wr \quad \text{Dold-Kan theorem}"] \\
\left[\text{chain complexes}\right] \arrow[d, "H_\bullet"', "\text{take the homology}"] \\
\left[\text{graded abelian groups}\right]
\end{tikzcd}$$

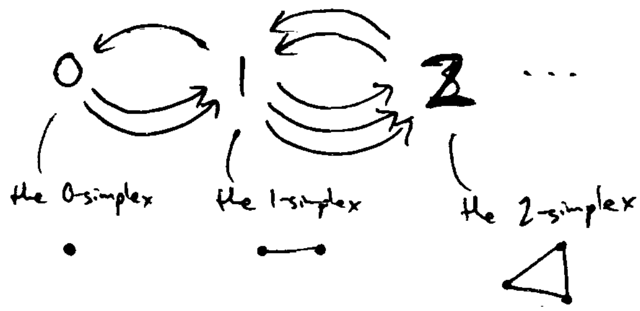
Just for now, let's write

$$\Delta = \Delta_{top}$$

So

$$\Delta = \begin{tikzcd}
1 \arrow[r, shift left=2] \arrow[r, shift right=2] & 2 \arrow[l] \arrow[r, shift left=3] \arrow[r] \arrow[r, shift right=3] & 3 \arrow[l, shift left=1.5] \arrow[l, shift right=1.5] & \cdots
\end{tikzcd}$$

But "n" corresponds to the (n-1)-simplex, so let's reindex and write:

$\Delta =$ 

For a topologist, a simplicial set is a functor

$$X : \Delta^{op} \longrightarrow Set$$

i.e. a diagram like

$$\begin{tikzcd} X(0) \arrow[r, bend left, "s_0"] & X(1) \arrow[l, shift left, "\partial_0"] \arrow[l, bend left=60, "\partial_1"] \arrow[r, bend left=50, "s_0"] \arrow[r, bend left=20, "s_1"] & X(2) \arrow[l, shift left, "\partial_0"] \arrow[l, bend left=30, "\partial_1"] \arrow[l, bend left=60, "\partial_2"] \cdots \end{tikzcd}$$

in Set.

To apply linear algebra, we take formal $\mathbb{Z}$-linear combinations of simplices & get abelian groups $FX(n)$ where

$$F: Set \longrightarrow AbGp$$

is the free abelian group functor.

So we get

$$FX : \Delta^{op} \longrightarrow AbGp$$

i.e. a diagram like

$$\begin{tikzcd}
FX(0) \arrow[r, bend left, "F(\iota_0)"] & FX(1) \arrow[l, bend left=10, "F(\partial_0)"'] \arrow[l, bend left=40, "F(\partial_1)"] \arrow[r, bend left=20] \arrow[r, bend left=40] & FX(2) \arrow[l, bend left=10] \arrow[l, bend left=25] \arrow[l, bend left=45]
\end{tikzcd}$$

in AbGp.

In general, for any category C, we call a functor

$$G : \Delta^{op} \longrightarrow C$$

a <u>simplicial object</u> in C. In particular, we call a functor

$$\Delta^{op} \longrightarrow AbGp$$

a simplicial abelian group. In fact, a simplicial abelian group is just a <u>chain complex</u> of abelian groups: a sequence of abelian group homomorphisms

$$C_0 \xleftarrow{\partial} C_1 \xleftarrow{\partial} C_2 \xleftarrow{\partial} \cdots$$

such that $\partial^2 = 0$.

How do we turn a simplicial abelian group

$$G : \Delta^{op} \longrightarrow AbGp$$

into a chain complex $Ch(G)$:

$$Ch(G)_0 \xleftarrow{\partial} Ch(G)_1 \xleftarrow{\partial} Ch(G)_2 \xleftarrow{\partial} \cdots \quad ?$$

Here's how: let

$$Ch(G)_n = G(n) / \operatorname{im} \iota_0 + \cdots + \operatorname{im} \iota_{n-1}$$

where the ι's are degeneracies:

$$\begin{tikzcd} G(0) \arrow[r, "\iota_0"] & G(1) \arrow[r, shift left, "\iota_0"] \arrow[r, shift right, "\iota_1"'] & G(2) \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \cdots \end{tikzcd}$$

This lets us ignore degenerate simplices. Then, let

$$\partial : Ch(G)_n \longrightarrow Ch(G)_{n-1}$$

be given by

$$\partial = \sum_{i=0}^{n} (-1)^i \partial_i$$

You need to check that:

1) ∂ is well defined as a map $Ch(G)_n \longrightarrow Ch(G)_{n-1}$.
(Check if $x \in G_n$ is degenerate, $\partial x = 0$ mod degenerate simplices)

2) $\partial^2 = 0$.

For example:

$$\begin{tikzcd} & y \arrow[dr, "g"] & \\ x \arrow[ur, "f"] \arrow[rr, "h"'] & & z \end{tikzcd}$$

X is a simplicial set w/
$X(0) = \{x, y, z\}$
$X(1) = \{f, g, h, \iota_0(x), \iota_0(y), \iota_0(z)\}$
$X(2) = \{\text{degenerate 2-simplices}\}$
$\vdots \qquad \vdots$

This gives a simplicial abelian group $G = FX$ with

$$G(0) = F\{x, y, z\}$$
$$G(1) = F\{f, g, h, \iota_0(x), \iota_0(y), \iota_0(z)\}$$
$$G(2) = F\{\text{degenerate 2-simplices}\}$$

This gives a chain complex $Ch(G)$ with

$$Ch(G)_0 = F\{x, y, z\}$$
$$Ch(G)_1 = F\{f, g, h\}$$
$$Ch(G)_2 = \{0\}$$

and

$$\partial : Ch(G)_1 \longrightarrow Ch(G)_0$$

is given by

$$\partial f = \partial_0 f - \partial_1 f = y - x$$
$$\partial g = \partial_0 g - \partial_1 g = z - y$$
$$\partial h = \partial_0 h - \partial_1 h = z - x$$

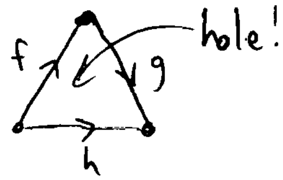
Here $\partial^2 = 0$ for trivial reasons. But there is still something interesting: $\partial^2 = 0$ means

$$\mathrm{im}(\partial : C_{n+1} \to C_n) \subseteq \ker(\partial : C_n \to C_{n-1})$$

— if something is a boundary, it has vanishing boundary of its own. But not vice versa, since there can be "holes". So we can keep track of holes using the homology groups

$$H_n(C) = \frac{\ker(\partial : C_n \longrightarrow C_{n-1})}{\mathrm{im}(\partial : C_{n+1} \longrightarrow C_n)}$$

Our example has a hole:



So compare

$$\ker(\partial : C_1 \to C_0) = F\{f+g-h\}$$

to

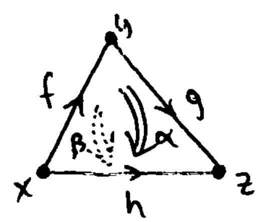
$$\mathrm{im}(\partial : C_2 \to C_1) = \{0\}$$

so

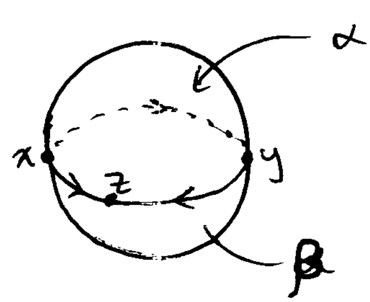
$$H_1(C) = \frac{F\{f+g-h\}}{\{0\}} \cong \mathbb{Z}$$

We're getting the free abelian group on <u>one</u> generator, indicating that we have <u>one</u> "hole".

Now let's do a less trivial example by making X bigger. Our new X will look like this:



α, β are two 2-simplices filling in the same hole. This gives a higher dimensional hole:



We calculate:

$$\partial \alpha = f + g - h$$
$$\partial \beta = f + g - h$$

So now

$$\mathrm{im}(\partial : C_2 \to C_1) = F\{f+g-h\}$$
$$\ker(\partial : C_1 \to C_0) = F\{f+g-h\}$$

so we still have $\mathrm{im} \subseteq \ker$ here, so

$$C_2 \xrightarrow{\partial} C_1 \xrightarrow{\partial} C_0$$

is zero: $\partial^2 = 0$. We also see that

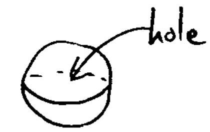
$$H_1(C) = \{0\}$$

– we've filled the hole in the triangle. But

$$\mathrm{im}(\partial : C_3 \to C_2) = \{0\}$$
$$\mathrm{im}(\partial : C_2 \to C_1) = F\{\alpha - \beta\}$$

so we get

$$H_2(C) = \frac{F\{\alpha-\beta\}}{\{0\}} \cong \mathbb{Z}$$

So there's one hole, but now of a higher dimension: 

$\alpha - \beta$ describes the surface of this 2-sphere, which has vanishing boundary but is not itself a boundary