Smooth Functors vs. Connections on Principal Bundles

We're generalizing the idea of a smooth functor

$$e^{iS} : PM \longrightarrow U(1)$$

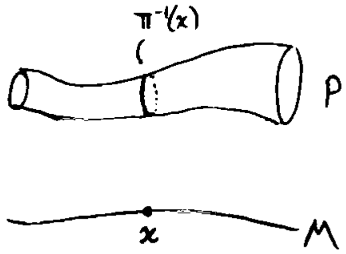
to the concept of "parallel transport in a principal $U(1)$-bundle"

Def: For any Lie group G, a principal G-bundle over the smooth space M is a smooth space P equipped with a (smooth) right action of G

$$\alpha : P \times G \longrightarrow P$$

and a map

$$\begin{array}{c} P \\ \pi \downarrow \\ M \end{array}$$



such that:

1. the action α preserves the fibers $\pi^{-1}(x)$ $\forall x \in M$ (i.e. $\forall p \in \pi^{-1}(x)$, $g \in G$ we have $\alpha(p,g) \in \pi^{-1}(x)$)
2. $\pi : P \longrightarrow M$ is locally trivial: $\forall x \in M$ $\exists$ open $U \ni x$ and an isomorphism of right G-spaces

$\gamma : \pi^{-1}(U) \longrightarrow U \times G$ s.t.

$$\begin{tikzcd} \pi^{-1}(U) \arrow[rr, "\gamma"] \arrow[dr] & & U \times G \arrow[dl] \\ & U & \end{tikzcd}$$

commutes.

Note: this implies that every fiber $P_x = \pi^{-1}(x)$ is a right G-space, but in fact we have an iso. of right G-spaces

$$\gamma : P_x \longrightarrow \{x\} \times G \cong G$$

so P_x is a G-torsor!

Now let's talk about parallel transport — a new way. Given a principal G-bundle $\pi : P \longrightarrow M$, we can form the <u>transport groupoid</u> $\mathrm{Trans}(P)$, where:

- objects are points $x \in M$ (or if you prefer, fibers P_x)
- morphisms $f : x \longrightarrow y$ are maps (hence isomorphisms) between the torsors P_x & P_y.

a G-set map between torsors is an isomorphism.

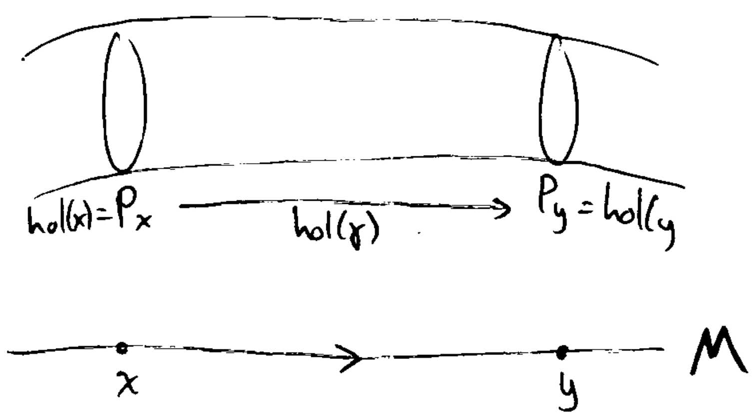
When $G = U(1)$, we think of P as a "circle bundle" over M, giving one circle of possible phases P_x for each $x \in M$. $\mathrm{Trans}(P)$ has these circles as objects & "rotations" from one circle to another as morphisms.

Def: A connection on P is a smooth functor

$$\mathrm{hol} : PM \longrightarrow \mathrm{Trans}(P)$$

where PM is the smooth category (groupoid!) of equivalence classes of paths in M, such that

$$\mathrm{hol}(x) = P_x \qquad \forall x \in M.$$



We can think of any smooth space M as a smooth category with:

- points of M as objects $\quad \mathrm{Ob}(M) = M$
- only identity morphisms $\quad \mathrm{Mor}(M) = M$

Then the condition $hol(x) = P_x$ says:

$$\begin{tikzcd} PM \arrow[r, "hol"] & \mathrm{Trans}(P) \\ & M \arrow[ul] \arrow[u] \end{tikzcd}$$ commutes.

Let's examine this locally: For any $x \in M$, there's an open set $U \ni x$ & an iso γ s.t.

$$\begin{tikzcd} P|_U := \pi^{-1}(U) \arrow[rr, "\gamma", "\sim"'] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$ commutes.

The inclusion $U \hookrightarrow M$ gives a smooth functor

$$PU \rightarrowtail PM$$

and a smooth functor

$$\mathrm{Trans}(P|_U) \rightarrowtail \mathrm{Trans}(P)$$

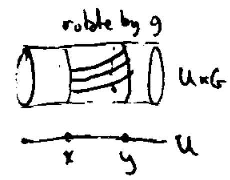
In fact we get a commuting prism

$$\begin{tikzcd}
 & & PM \arrow[r, "\mathrm{hol}"] & \mathrm{Trans}(P) \\
PU \arrow[urr] \arrow[r, "\mathrm{hol}|_U"'] & \mathrm{Trans}(P|_U) \arrow[urr] & M \arrow[u, dotted] \arrow[ur] & \\
 & U \arrow[ul] \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

for a unique $\mathrm{hol}|_U : PU \longrightarrow \mathrm{Trans}(P|_U)$. So we can "restrict parallel transport to U" But we have

$$\begin{tikzcd}
P|_U \arrow[rr, "\gamma", "\sim"'] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\pi_1"] \\
 & U &
\end{tikzcd}$$

So $\mathrm{Trans}(P|_U) \cong \mathrm{Trans}(U \times G)$ where $U \times G$ is the trivial principal G-bundle over U. But



$$\mathrm{Ob}(\mathrm{Trans}(U \times G)) \cong U$$
$$\mathrm{Mor}(\mathrm{Trans}(U \times G)) \cong U^2 \times G$$

and in fact
$$\mathrm{Trans}(U \times G) \cong \mathrm{Codisc}(U) \times G$$

where $\mathrm{Codisc}(U)$ has elts $x \in U$ as objects and 1 morphism from any object to any other, & G is regarded as a 1-object category. So we get

$$PU \xrightarrow{\mathrm{hol}|_U} \mathrm{Trans}(P|_U) \cong \mathrm{Trans}(U \times G) \cong \mathrm{Codisc}(U) \times G \xrightarrow{\pi_2} G$$

a smooth functor from PU to G which contains all the information in $\mathrm{hol}|_U$. But...

Thm: Smooth functors $\mathrm{hol}: PU \to G$ are in 1-1 correspondence with $\mathfrak{g}$-valued 1-forms, where $\mathfrak{g}$ is the Lie algebra of G.

So connections on principal G-bundles are locally described by $\mathfrak{g}$-valued 1-forms. For $G = U(1)$, these are just 1-forms.