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Simplicial Sets from Algebraic Gadgets

To count "holes" in any algebraic gadget, we'll describe how to get a simplicial set from it. Then from a simplicial set we can (freely) form a simplicial abelian group, & thus a chain complex, whose homology "counts the holes" in our simplicial set.

Any algebraic gadget lives in a category D that's related to some underlying category C by a pair of adjoint functors

$$\begin{tikzcd} D \arrow[d, shift left, "R"] \\ C \arrow[u, shift left, "L"] \end{tikzcd}$$

free functor — L D algebraic gadgets R — forgetful functor,

For example:

$$\begin{tikzcd} \mathrm{AbGp} \arrow[d, shift left, "R"] \\ \mathrm{Set} \arrow[u, shift left, "L"] \end{tikzcd}$$

If $A \in \mathrm{AbGp}$, RA is its underlying set

If $S \in \mathrm{Set}$, LS is the free abelian gp. on S.

In general objects in D should be thought of as objects of C "equipped with extra data" – R forgets these data, L freely generates the data.

More formally we require that

$$\begin{tikzcd} D \arrow[d, shift left, "R"] \\ C \arrow[u, shift left, "L"] \end{tikzcd}$$

is an <u>adjunction</u>, with L as left adjoint of R, R as right adjoint of L, meaning there's a natural isomorphism

$$\hom(Lc, d) \cong \hom(c, Rd) \qquad \begin{array}{l} \forall c \in C \\ \forall d \in D \end{array}$$

For example: with

$$\begin{tikzcd} \mathrm{AbGp} \arrow[d, shift left, "R"] \\ \mathrm{Set} \arrow[u, shift left, "L"] \end{tikzcd}$$

given any set S & abelian gp. A

$$\hom(LS, A) \cong \hom(S, RA)$$

so there's a natural 1-1 correspondence between abelian group homomorphisms $f : LS \longrightarrow A$
and functions $\tilde{f} : S \longrightarrow RA$.

We'll construct our simplicial set using the 'unit' and 'counit' of the adjunction

$$\begin{tikzcd} D \arrow[d, shift left, "R"] \\ C \arrow[u, shift left, "L"] \end{tikzcd}$$

Starting with

$$\hom(Lc, d) \cong \hom(c, Rd)$$

we can take $d = Lc$, so

$$\begin{aligned} \hom(Lc, Lc) &\cong \hom(c, RLc) \\ 1_c &\longmapsto \iota_c \end{aligned}$$

where

$$\iota_c : c \longrightarrow RLc$$

is a morphism called the <u>unit</u> of $c \in C$. Similarly, we can take $c = Rd$, so

$$\begin{aligned} \hom(LRd, d) &\cong \hom(Rd, Rd) \\ \varepsilon_d &\longleftarrow\!\shortmid 1_d \end{aligned}$$

where

$$\varepsilon_d : LRd \longrightarrow d$$

is a morphism called the <u>counit</u> of $d \in D$.

In fact, η & ε have very nice meanings, as seen in examples, e.g.

$$\begin{tikzcd} \mathrm{AbGp} \arrow[d, shift left, "R"] \\ \mathrm{Set} \arrow[u, shift left, "L"] \end{tikzcd}$$

Here the unit is a function

$$\eta_S : S \longrightarrow \underbrace{RLS}$$

set of elements of the free abelian group on S

which (check it!) is just the "inclusion of the generators".

The counit is an abelian group homomorphism

$$\varepsilon_A : LRA \longrightarrow A$$

sending any formal $\mathbb{Z}$-linear combination

$$\sum_i n_i (a_i) \qquad \begin{array}{l} n_i \in \mathbb{Z} \\ a_i \in A \end{array}$$

where $(a) \in RA$ is the element of RA corresponding to $a \in A$, to the actual linear combination

$$\sum_i n_i a_i \ \in A .$$

In this example, $\varepsilon_A : LRA \to A$ is always onto, so

$$A = \frac{LRA}{\ker \varepsilon_A}$$

In other words, we have a presentation of A with RA as the set of generators & $\ker \varepsilon_A$ as relations. This is the "canonical presentation" of A — and this idea works for almost any algebraic gadget. Soon we'll see "relations between relations" etc.

The moral:

the unit η "includes the generators"
the counit ε "imposes the relations", or
"maps formal expressions (in LRA) to actual expressions (in A)"
or "evaluates formal expressions"

Now we'll use η & ε to build a simplicial object in D from any object $d \in D$!

$$\begin{tikzcd}
d & LRd \arrow[l, "\varepsilon_d"'] \arrow[r, bend left, "L(\eta_{Rd})"] & LRLRd \arrow[l, bend left=20, "\varepsilon_{LRd}"'] \arrow[l, bend left=50, "LR(\varepsilon_d)"] \arrow[r, bend left=20, "LRL(\eta_{Rd})"] \arrow[r, bend left=50, "L(\eta_{RLRd})"] & LRLRLRd \arrow[l, bend left=15, "\varepsilon_{LRLRd}"'] \arrow[l, bend left=35, "LR\varepsilon_{LRd}"] \arrow[l, bend left=60, "LRLR\varepsilon_d"] & \cdots
\end{tikzcd}$$

$$\begin{tikzcd}
D \arrow[d, shift left, "R"] \\
C \arrow[u, shift left, "L"]
\end{tikzcd}$$

$\eta_c : c \to RLc$
$\varepsilon_d : LRd \to d$

Next time we'll check that this gives an "algebraist's simplicial object". Recall:

$$\Delta_{alg} \sim \begin{tikzcd} 0 \arrow[r] & 1 \arrow[r, shift left] \arrow[r, shift right] & 2 \arrow[l, bend right] \arrow[r] \arrow[r, shift left=2] \arrow[r, shift right=2] & 3 \arrow[l, bend right, shift right] \arrow[l, bend right, shift left] \end{tikzcd}$$

& an algebraist's simplicial object in D is a functor

$$F : \Delta_{alg}^{op} \longrightarrow D.$$

This is called the <u>bar construction</u> – developed by Eilenberg and Mac Lane (in a special case).

Let's consider

$$\begin{tikzcd} \mathrm{AbGp} \arrow[d, shift left, "R"] \\ \mathrm{Set} \arrow[u, shift left, "L"] \end{tikzcd}$$

Take $\mathbb{Z} \in \mathrm{AbGp}$ & do the bar construction to it.

A typical element of $\mathbb{Z}$ looks like

$$211.$$

A typical element of $LR\mathbb{Z}$ looks like

$$(3) + (11) - 2(6) \quad := (3) + (11) - (6) - (6).$$

Apply $\varepsilon_{\mathbb{Z}} : LR\mathbb{Z} \longrightarrow \mathbb{Z}$ we get

$$\varepsilon_{\mathbb{Z}}\big((3) + (11) - 2(6)\big) = 3 + 11 - 2\cdot 6 = 2$$

The counit "strips off parentheses". A typical element of $LRLR\mathbb{Z}$ looks like

$$((2)+(5)) - ((4)-(1)) + ((1))$$

Applying $\varepsilon_{LR\mathbb{Z}} : LRLR\mathbb{Z} \longrightarrow LR\mathbb{Z}$ we get

$$\begin{aligned}\varepsilon_{LR\mathbb{Z}}\Big(((2)+(5)) - ((4)-(1)) + ((1))\Big) \\ &= (2)+(5)-(4)+(1)+(1) \\ &= (2)+(5)-(4)+2(1)\end{aligned}$$

Applying $LR(\varepsilon_{\mathbb{Z}}) : LRLR\mathbb{Z} \longrightarrow LR\mathbb{Z}$ we get

$$\begin{aligned}LR(\varepsilon_{\mathbb{Z}})\Big(((2)+(5)) - ((4)-(1)) + ((1))\Big) \\ &= (7) - (3) + (1)\end{aligned}$$

You could apply $\varepsilon_{\mathbb{Z}}$ to either $(2)+(5)-(4)+(1)$ or $(7)-(3)+(1)$ & get the same answer: $5 \in \mathbb{Z}$.

Really we're looking at a 1-simplex in our simplicial abelian group

$$(7)-(3)+(1) \xrightarrow{((2)+(5))-((4)-(1))+(1)} (2)+(5)-(4)+2(1)$$

think of this as a proof that

$$7-3+1 = 2+5-4+2\cdot 1$$