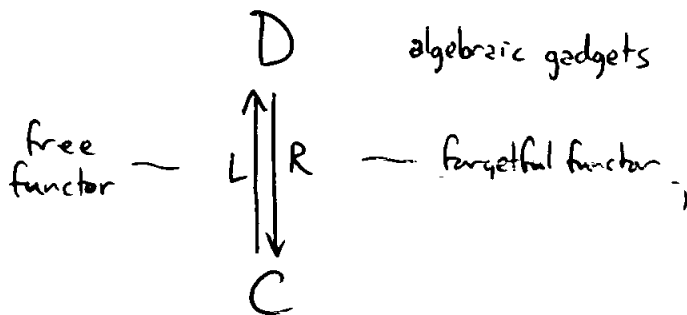


10 May 2007

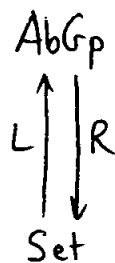
Simplicial Sets from Algebraic Gadgets

To count "holes" in any algebraic gadget, we'll describe how to get a simplicial set from it. Then from a simplicial set we can (freely) form a simplicial abelian group, & thus a chain complex, whose homology "counts the holes" in our simplicial set.

Any algebraic gadget lives in a category D that's related to some underlying category C by a pair of adjoint functors



For example:



If $A \in \text{AbGp}$, RA is its underlying set

If $S \in \text{Set}$, LS is the free abelian gp. on S .

In general objects in D should be thought of as objects of C "equipped with extra data" - R forgets these data, L freely generates the data.

More formally we require that

$$\begin{array}{c} D \\ \uparrow L \downarrow R \\ C \end{array}$$

is an adjunction, with L as left adjoint of R , R as right adjoint of L , meaning there's a natural isomorphism

$$\text{hom}(Lc, d) \cong \text{hom}(c, Rd) \quad \begin{array}{l} \forall c \in C \\ \forall d \in D \end{array}$$

For example: with

$$\begin{array}{c} \text{AbGp} \\ \uparrow L \downarrow R \\ \text{Set} \end{array}$$

given any set S & abelian gp. A

$$\text{hom}(LS, A) \cong \text{hom}(S, RA)$$

so there's a natural 1-1 correspondence between abelian group homomorphisms $f: LS \rightarrow A$

and functions $\tilde{f}: S \rightarrow RA$.

We'll construct our simplicial set using the 'unit' and 'counit' of the adjunction

$$\begin{array}{c} D \\ \uparrow L \quad \downarrow R \\ C \end{array}$$

Starting with

$$\text{hom}(Lc, d) \cong \text{hom}(c, Rd)$$

we can take $d = Lc$, so

$$\text{hom}(Lc, Lc) \cong \text{hom}(c, RLc)$$

$$1_c \longmapsto \eta_c$$

where

$$\eta_c: c \rightarrow RLc$$

is a morphism called the unit of $c \in C$. Similarly, we can take $c = Rd$, so

$$\text{hom}(LRd, d) \cong \text{hom}(Rd, Rd)$$

$$\epsilon_d \longleftarrow 1_d$$

where

$$\epsilon_d: LRd \rightarrow d$$

is a morphism called the counit of $d \in D$.

In fact, ι & ε have very nice meanings, as seen in examples, e.g.

$$\begin{array}{c} \text{AbGp} \\ \uparrow \downarrow \\ \text{Set} \end{array}$$

Here the unit is a function

$$\iota_S : S \longrightarrow \text{RLS}$$

↳ set of elements
of the free abelian
group on S

which (check it!) is just the "inclusion of the generators".

The counit is an abelian group homomorphism

$$\varepsilon_A : \text{LRA} \longrightarrow A$$

sending any formal \mathbb{Z} -linear combination

$$\sum_i n_i (a_i) \quad \begin{array}{l} n_i \in \mathbb{Z} \\ a_i \in A \end{array}$$

where $(a) \in \text{RA}$ is the element of RA corresponding to $a \in A$, to the actual linear combination

$$\sum_i n_i a_i \in A.$$

In this example, $\epsilon_A: LRA \rightarrow A$ is always onto, so

$$A = \frac{LRA}{\ker \epsilon_A}$$

In other words, we have a presentation of A with RA as the set of generators & $\ker \epsilon_A$ as relations.

This is the "canonical presentation" of A — and this idea works for almost any algebraic gadget. Soon we'll see "relations between relations" etc.

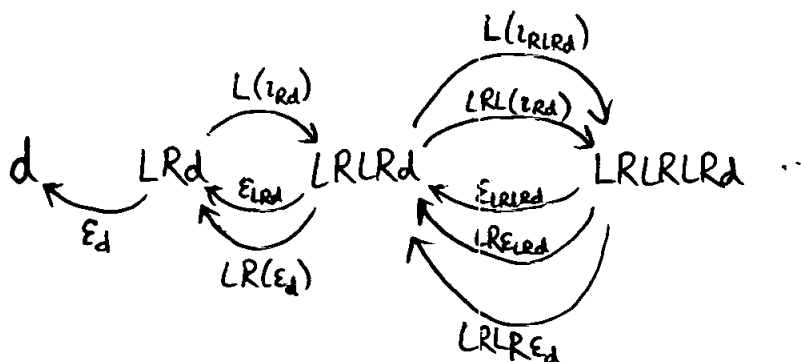
The moral:

the unit ι "includes the generators"

the counit ϵ "imposes the relations", or

"maps formal expressions (in LRA) to actual expressions (in A)"
or "evaluates formal expressions"

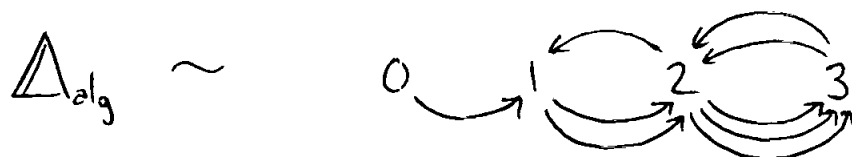
Now we'll use ι & ϵ to build a simplicial object in D from any object $d \in D$!



$$\begin{array}{c} D \\ \uparrow \downarrow \\ L \quad R \\ \downarrow \uparrow \\ C \end{array}$$

$\iota_c: c \rightarrow Rc$
 $\epsilon_d: LRd \rightarrow d$

Next time we'll check that this gives an "algebraist's simplicial object". Recall:



& an algebraist's simplicial object in D is a functor

$$F: \Delta_{\text{alg}}^{\text{op}} \rightarrow D.$$

This is called the bar construction — developed by Eilenberg and Mac Lane (in a special case).

Let's consider

$$\begin{array}{c} \text{AbGp} \\ L \uparrow \downarrow R \\ \text{Set} \end{array}$$

Take $\mathbb{Z} \in \text{AbGp}$ & do the bar construction to it.

A typical element of \mathbb{Z} looks like

$$211.$$

A typical element of $LR\mathbb{Z}$ looks like

$$(3) + (11) - 2(6) := (3) + (11) - (6) - (6).$$

Apply $\varepsilon_{\mathbb{Z}} : LR\mathbb{Z} \rightarrow \mathbb{Z}$ we get

$$\varepsilon_{\mathbb{Z}}((3) + (11) - 2(6)) = 3 + 11 - 2 \cdot 6 = 2$$

The counit "strips off parentheses". A typical element of $LRLR\mathbb{Z}$ looks like

$$((2) + (5)) - ((4) - (1)) + ((1))$$

Applying $\varepsilon_{LR\mathbb{Z}} : LRLR\mathbb{Z} \rightarrow LR\mathbb{Z}$ we get

$$\begin{aligned} \varepsilon_{LR\mathbb{Z}}\left(\left(\left(2\right) + \left(5\right)\right) - \left(\left(4\right) - \left(1\right)\right) + \left(\left(1\right)\right)\right) \\ = (2) + (5) - (4) + (1) + (1) \\ = (2) + (5) - (4) + 2(1) \end{aligned}$$

Applying $LR(\varepsilon_{\mathbb{Z}}) : LRLR\mathbb{Z} \rightarrow LR\mathbb{Z}$ we get

$$\begin{aligned} LR(\varepsilon_{\mathbb{Z}})\left(\left(\left(2\right) + \left(5\right)\right) - \left(\left(4\right) - \left(1\right)\right) + \left(\left(1\right)\right)\right) \\ = (7) - (3) + (1) \end{aligned}$$

You could apply $\varepsilon_{\mathbb{Z}}$ to either $(2) + (5) - (4) + (1)$ or $(7) - (3) + (1)$ & get the same answer: $5 \in \mathbb{Z}$.

Really we're looking at a 1-simplex in our simplicial abelian group

$$\begin{array}{ccc}
 & \xrightarrow{((2)+(5)) - ((4)-(1)) + (1)} & \\
 \bullet & \xrightarrow{\hspace{10em}} & \bullet \\
 (7) - (3) + (1) & & (2) + (5) - (4) + 2(1)
 \end{array}$$

think of this as a proof
that

$$7 - 3 + 1 = 2 + 5 - 4 + 2 \cdot 1$$