Connections & Smooth Anafunctors

Last time we described a connection on a principal G-bundle $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ as a smooth functor:

$$\mathrm{hol} : PM \longrightarrow \mathrm{Trans}(P)$$

such that

$$\mathrm{hol}(x) = P_x .$$

Last time we expressed this last clause by saying that

$$\begin{tikzcd} & \mathrm{Disc}(M) \arrow[dl] \arrow[dr] & \\ PM \arrow[rr, "\mathrm{hol}"'] & & \mathrm{Trans}(P) \end{tikzcd}$$

discrete category on M.

commutes.

i.e. we're looking at smooth categories and functors <u>under</u> Disc(M). A better way to express this clause is to say ~~PM hol~~

$$\begin{tikzcd} PM \arrow[rr] \arrow[dr] & & \mathrm{Trans}(P) \arrow[dl] \\ & \mathrm{Codisc}(M) & \end{tikzcd}$$

commutes

codiscrete category on M.

Okay... now for a big chart...

	Connections	Gauge Transformations

Trivial G-bundles

$$\begin{array}{c} M\times G \\ \downarrow \\ M \end{array}$$

Connections: smooth functors

$$hol : PM \longrightarrow G$$

equivalently: $\mathfrak{g}$-valued 1-forms on M

Gauge Transformations: smooth natural transformations

$$\begin{tikzcd} PM \arrow[r, bend left, "hol_1"] \arrow[r, bend right, "hol_2"'] \arrow[r, phantom, "\Downarrow g"] & G \end{tikzcd}$$

equivalently: smooth functions

$$g : M \longrightarrow G$$

(s.t. $hol_2(\gamma) = g_y \, hol_1(\gamma) g_x^{-1}$ for $\gamma : x \to y$ in PM)

Any fixed G-bundle

$$\begin{array}{c} P \\ \downarrow \\ M \end{array}$$

Connections: smooth functors

$$\begin{tikzcd} PM \arrow[rr, "hol"] \arrow[dr] & & Trans(P) \arrow[dl] \\ & Codisc(M) & \end{tikzcd}$$

Gauge Transformations: smooth natural transformations

$$\begin{tikzcd} PM \arrow[rr, bend left, "hol_1"] \arrow[rr, bend right, "hol_2"'] \arrow[rr, phantom, "\Downarrow g"] \arrow[dr] & & Trans(P) \arrow[dl] \\ & Codisc(M) & \end{tikzcd}$$

A variable G-bundle over M

Connections: smooth anafunctors

$$hol : PM \longrightarrow G$$

Gauge Transformations: smooth ananatural transformations

$$\begin{tikzcd} PM \arrow[r, bend left, "hol_1"] \arrow[r, bend right, "hol_2"'] \arrow[r, phantom, "\Downarrow g"] & G \end{tikzcd}$$

Remarks:

1) Connections mod gauge transformations are classified by various forms of cohomology. For example

$$\{U(1) \text{ bundles w. conn. over } M\} / \{\text{gauge trans.}\}$$

is an example of "Deligne cohomology."

2) To get a smooth functor, "parallel transport"

$$\mathrm{hol} : PM \longrightarrow G$$

from a $\mathfrak{g}$-valued 1-form A on M we set

$$\mathrm{hol}(\gamma) = P e^{\int_\gamma A}$$

If $G = U(1)$ then $\mathfrak{g} = \mathfrak{u}(1) = i\mathbb{R} \cong \mathbb{R}$ so a $\mathfrak{g}$-valued 1-form amounts to a 1-form A and then

$$\mathrm{hol}(\gamma) = e^{i\int_\gamma A}$$

(Path ordered exponentiation reduces to ordinary exponentiation since $U(1)$ is abelian).

3) Given smooth functors

$$F_1, F_2 : C \longrightarrow D$$

a <u>smooth natural transformation</u> is a smooth map

$$\alpha : \mathrm{Ob}(C) \longrightarrow \mathrm{Mor}(D)$$

s.t. $\forall$ morphism $f : x \to y$ in C this square exists

$$\begin{tikzcd}
F_1(x) \arrow[r, "F_1(f)"] \arrow[d, "\alpha_x"'] & F_1(y) \arrow[d, "\alpha_y"] \\
F_2(x) \arrow[r, "F_2(f)"'] & F_2(y)
\end{tikzcd}$$

& commutes.

So a gauge transformation

$$\begin{tikzcd} PM \arrow[r, bend left, "hol_1"] \arrow[r, bend right, "hol_2"'] \arrow[r, phantom, "\Downarrow"] & G \end{tikzcd}$$

is a smooth natural transformation from hol_1 to hol_2, i.e. a smooth map

$$g : Ob(PM) \longrightarrow Mor(G)$$

i.e.

$$g : M \longrightarrow G$$

such that given a path $\gamma : x \longrightarrow y$ in M, this square commutes:

$$\begin{tikzcd} * \arrow[r, "hol_1(\gamma)"] \arrow[d, "g_x"'] & * \arrow[d, "g_y"] \\ * \arrow[r, "hol_2(\gamma)"'] & * \end{tikzcd}$$

where $*$ is the one object of our group G. This says

$$hol_2(\gamma) = g_y \, hol_1(\gamma) \, g_x^{-1} .$$

4) Given any G-bundle $\begin{matrix} P \\ \downarrow \\ M \end{matrix}$ and two connections

$$hol_1,\ hol_2 : PM \longrightarrow Trans(P)$$

a smooth natural transformation

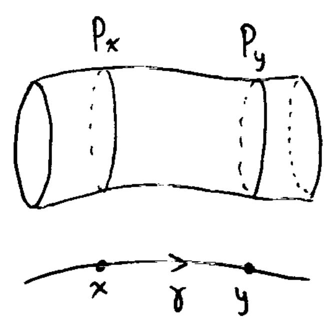
$$\begin{tikzcd} PM \arrow[r, bend left, "hol_1"] \arrow[r, bend right, "hol_2"'] \arrow[r, phantom, "\Downarrow g"] & Trans(P) \end{tikzcd}$$

is a smooth map

$$g : \underbrace{Ob(PM)}_{\overset{\shortparallel}{M}} \longrightarrow Mor(Trans(P))$$

s.t. this square commutes

$$\begin{tikzcd} P_x \arrow[r, "hol_1(\gamma)"] \arrow[d, "g_x"'] & P_y \arrow[d, "g_y"] \\ P_x \arrow[r, "hol_2(\gamma)"'] & P_y \end{tikzcd}$$



where $g_x : P_x \longrightarrow P_x$ is a G-torsor morphism

$$g_x(ph) = g_x(p)h \qquad \begin{matrix} \forall p \in P_x \\ \forall h \in G \end{matrix}$$

& similarly for g_y

5. What's a smooth anafunctor? Given smooth categories C & D a smooth anafunctor is the <u>right</u> kind of thing going from C to D, generalizing a smooth functor. A smooth anafunctor "looks locally like a smooth functor", so it can be thought of as a functor which is locally isomorphic (via natural isomorphisms) to a smooth one. More precisely:

<u>Def</u>: Let C, D be smooth categories. A <u>smooth anafunctor</u> $F: C \to D$ consists of:

1) an open cover $\{U_\alpha\}$ of $Ob(C)$, where $Ob(C)$ is a topological space with the finest topology s.t. every plot in $Ob(C)$ is continuous.

2) smooth functors

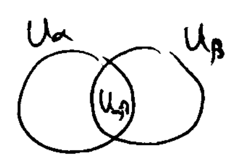
$$F_\alpha : C_\alpha \to D$$

where C_α has:

- objects in U_α as objects
- morphisms between these as morphisms.

3) smooth natural isomorphisms

$$\begin{tikzcd} & C_\alpha \arrow[dr, "F_\alpha"] \arrow[dd, phantom, "\Downarrow g_{\alpha\beta}"] & \\ C_{\alpha\beta} \arrow[ur] \arrow[dr] & & D \\ & C_\alpha \arrow[ur, "F_\beta"'] & \end{tikzcd}$$



where $C_{\alpha\beta}$ has:

- objects in $U_{\alpha\beta} = U_\alpha \cap U_\beta$ as objects
- morphisms between these as morphisms.

4) Finally require the "cocycle conditions":

$$g_{\alpha\beta}\, g_{\beta\gamma} = g_{\alpha\gamma}$$

on objects in $U_{\alpha\beta\gamma} = U_\alpha \cap U_\beta \cap U_\gamma$.