22 May 2007

Bundles, Connections, Cohomology, & Anafunctors

Last time JB claimed that if M is a smooth space (e.g. a manifold) & G a Lie group then principal G-bundles with connection over M correspond to smooth anafunctors

$$\mathrm{hol}: PM \longrightarrow G.$$

JB also claimed that isomorphisms between principal G-bundles w. connection correspond to smooth ananatural isomorphisms

$$\begin{tikzcd} PM \arrow[r, bend left, "\mathrm{hol}"] \arrow[r, bend right, "\mathrm{hol}'"'] \arrow[r, phantom, "\Downarrow g"] & G \end{tikzcd}.$$

If we leave out the connection we get a simpler version of this story. Principal G-bundles over M correspond to smooth anafunctors:

$$\mathrm{hol}: \mathrm{Disc}(M) \longrightarrow G$$

& isomorphisms between principal G-bundles correspond to smooth ananatural transformations

$$\begin{tikzcd} \mathrm{Disc}(M) \arrow[r, bend left, "\mathrm{hol}"] \arrow[r, bend right, "\mathrm{hol}'"'] \arrow[r, phantom, "\Downarrow g"] & G \end{tikzcd}.$$

Let's see why this simpler version works. Our ultimate goal is to show

$$\frac{\{\text{Princ. } G\text{-bundles over } M\}}{\{\text{isomorphisms}\}} \cong \frac{\{\text{smooth anafunctors } \mathrm{Disc}(M) \to G\}}{\{\text{ananatural transformations}\}}$$

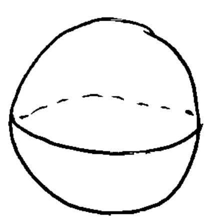
$$\cong \check{H}^1(M,G)$$

where $\check{H}^1(M,G)$ is the first Čech cohomology of M with coefficients in G. A famous example:

$$\check{H}^1(M, U(1)) \cong H^2(M, \mathbb{Z})$$

where $H^2(M,\mathbb{Z})$ is the second cohomology of M with coefficients in $\mathbb{Z}$.

E.g.



$M = S^2$

has $H^2(M,\mathbb{Z}) \cong \mathbb{Z}$

(S^2 has "one 2d hole")

So principal $U(1)$-bundles over S^2 are classified by an integer, the "first Chern class" c_1.

Suppose $\begin{array}{c} P \\ \downarrow \\ M \end{array}$ is a principal G-bundle. It's locally trivial: for each pt. $x \in M$ there's an open nbhd $U \ni x$ for which we have an isomorphism of G-spaces:

$$\begin{tikzcd} P|_U = \pi^{-1}(U) \arrow[rr, "\sim"] \arrow[dr, "\pi"'] & & U \times G \arrow[dl, "\pi_1"] \\ & U & \end{tikzcd}$$

Let's go ahead and pick an open cover $\{U_i\}$ of X and isomorphisms

$$\begin{tikzcd} P|_{U_i} \arrow[rr, "\gamma_i"] \arrow[dr] & & U_i \times G \arrow[dl] \\ & U_i & \end{tikzcd}$$

Let $U_{ij} = U_i \cap U_j$; we get "transition functions"

$$g_{ij} = \gamma_i \gamma_j^{-1} : U_{ij} \times G \xrightarrow{\sim} U_{ij} \times G$$

(here γ_i means γ_i restricted to $P|U_{ij}$; same for γ_j)

This is an isomorphism of right G-spaces, necessarily given by left multiplication by some G-valued function on U_{ij}.

By abuse of notation, we also call this function

$$g_{ij} : U_{ij} \longrightarrow G.$$

This is a "Čech 1-cochain": in general a Čech n-cochain would be a bunch of maps

$$g_{i_0 \cdots i_n} : \underset{\substack{\parallel \\ U_{i_0} \cap \cdots \cap U_{i_n}}}{U_{i_0 \cdots i_n}} \longrightarrow G$$

So: any principal G-bundle over M gives a Čech 1-cochain. In fact, g_{ij} satisfies an equation:

$$g_{ij} g_{jk} = \gamma_i \gamma_j^{-1} \gamma_j \gamma_k^{-1} = \gamma_i \gamma_k^{-1} = g_{ik}$$

on U_{ijk}. This is called the cocycle condition & we say g_{ij} is a <u>Čech 1-cocycle</u>. The cocycle condition really says that this 2-cochain:

$$g_{ij} g_{jk} g_{ik}^{-1} : U_{ijk} \longrightarrow G$$

is trivial, i.e. it equals 1.

Conversely, given a Čech 1-cocycle you can build a principal G-bundle over M.

So: "principal G-bundles correspond to Čech 1-cocycles".

What do isomorphisms of principal G-bundles correspond to?

Suppose P, P' are two principal G-bundles over M & we have an isomorphism

$$\begin{tikzcd} P \arrow[rr, "f"] \arrow[dr] & & P' \arrow[dl] \\ & M & \end{tikzcd}$$

(i.e. f is a smooth map of right G-spaces making the triangle commute, where we can think of M as a trivial G-space, so this diagram lives in the category of smooth G-spaces.) We can find an open cover U_i of M s.t. $P|_{U_i}$ & $P'|_{U_i}$ are trivializable:

$$U_i \times G \xrightarrow[\sim]{\gamma_i^{-1}} P|_{U_i} \xrightarrow[\sim]{f} P'|_{U_i} \xrightarrow{\gamma_i'} U_i \times G$$

where γ_i, γ_i' are trivializations — isos of right G-spaces over U_i. Let

$$f_i = \gamma_i' f \gamma_i^{-1}$$

be the composite; these f_i's describe our bundle iso. f locally; i.e. we can reconstruct $f: P|_{U_i} \to P'|_{U_i}$ from f_i, γ_i, γ_i'. Since $f_i: U_i \times G \to U_i \times G$ is an isomorphism of G-bundles, it's determined by a function from U_i to G, which we also call f_i: we can write

$$f_i(x,g)$$

as

$$(x, f_i(x)g)$$

$\exists f_i : U \to G$. Recall a bunch of maps $f_i : U \to G$ is called a <u>Čech 0-cochain</u>. The bundle P is described by a Čech 1-cochain:

$$g_{ij} := \gamma_i \gamma_j^{-1}$$

& similarly for P':

$$g'_{ij} := \gamma'_i \gamma'^{-1}_j$$

where

$$g_{ij}, g'_{ij} : U_{ij} \longrightarrow G$$

In fact, g_{ij} & g'_{ij} are related by f_i as follows:

$$f_i g_{ij} = g'_{ij} f_j$$

Let's čeck this:

$$\gamma'_i f \gamma_i^{-1} \gamma_i \gamma_j^{-1} \overset{?}{=} \gamma'_i \gamma'^{-1}_j \gamma'_j f \gamma_j^{-1}$$

$$\gamma'_i f \gamma_j^{-1} = \gamma'_i f \gamma_j^{-1} \quad \checkmark .$$

In short, the "difference" of the two Čech 1-cochains (actually 1-cocycles) g_{ij} & g'_{ij} is given by the Čech 0-cochain f_i — but if G is nonabelian we <u>can't</u> actually write $g_{ij} g'^{-1}_{ij} = f_j f_i^{-1}$. This works only if G is abelian. We're studying "nonabelian Čech cohomology"!

So:

$$\frac{\{G\text{-bundles } P \to M \text{ s.t. } P|_{U_i} \text{ are trivial}\}}{\text{isomorphism}} \cong \frac{\{\text{Čech 1-cocycles } g_{ij} : U_{ij} \to G\}}{g_{ij} \sim g'_{ij} \text{ iff } f_i g_{ij} = g'_{ij} f_j} .$$

Taking the "limit" (really colimit!) as the cover U_i gets finer, we get

$$\frac{\{G\text{-bundles } P \to M\}}{\text{isomorphism}} = \varinjlim_{U} \frac{\{\text{Čech 1-cocycles } g_{ij} : U_{ij} \to G\}}{\sim}$$

$$\| $$

$$\check{H}^1(M, G)$$

the 1st Čech cohomology of M with coefficients in G.

There's another description of this: (due to Toby Bartels)

$$\check{H}^1(M, G) = \frac{\{\text{smooth anafunctors } F : \mathrm{Disc}(M) \to G\}}{\text{ananatural isomorphisms}}$$