24 May 2007

The Bar Construction

Again, the question is: why do adjunctions give simplicial objects? We began to see the answer

1) Δ_{alg} is the "walking monoid" – the free monoidal category on a monoid.

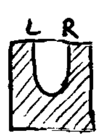
2) Any adjunction gives a monoid.

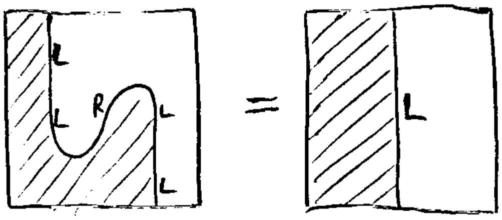
Let's continue with part 2). Suppose we have a pair of adjoint functors:

$$\begin{array}{c} D \\ L\uparrow\downarrow R \\ C \end{array}$$

Then we get natural transformation

the <u>unit</u>  $\iota : 1_C \Rightarrow RL$ (think "inclusion of the generators")

the <u>counit</u>  $\varepsilon : LR \Rightarrow 1_D$ (think "evaluation of formal expressions) and these – one can show – satisfy the "zig-zag identities":



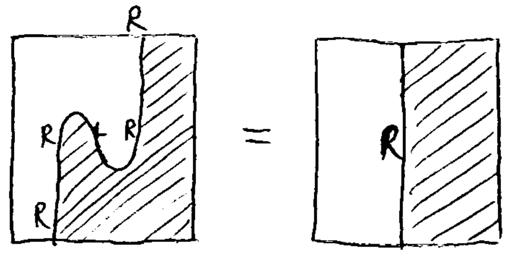
i.e.

$$\begin{tikzcd} L \arrow[d, Rightarrow, "L\iota"] \\ LRL \arrow[d, Rightarrow, "\varepsilon L"] \\ L \end{tikzcd} \quad = \quad \begin{tikzcd} L \arrow[dd, Rightarrow, "1_L"] \\ \\ L \end{tikzcd}$$

i.e. for each object $c \in C$

$$\begin{tikzcd} Lc \arrow[d, "L(\iota_c)"] \\ LRLc \arrow[d, "\varepsilon_{Lc}"] \\ Lc \end{tikzcd} \quad = \quad \begin{tikzcd} Lc \arrow[dd, "1_{Lc}"] \\ \\ Lc \end{tikzcd}$$

$\iota_c : c \to RLc$
$\varepsilon_d : LRd \to d$



i.e.

$$\begin{tikzcd} R \arrow[d, Rightarrow, "\iota R"] \\ RLR \arrow[d, Rightarrow, "R\varepsilon"] \\ R \end{tikzcd} \quad = \quad \begin{tikzcd} R \arrow[dd, Rightarrow, "1_R"] \\ \\ R \end{tikzcd}$$

i.e. for each object $d \in D$

$$\begin{tikzcd} Rd \arrow[d, "\iota_{Rd}"] \\ RLRd \arrow[d, "R(\varepsilon_d)"] \\ Rd \end{tikzcd} \quad = \quad \begin{tikzcd} Rd \arrow[dd, "1_{Rd}"] \\ \\ Rd \end{tikzcd}$$

We could prove  =  using the fact that

$$\hom(Lc, d) \cong \hom(c, Rd)$$

is a natural isomorphism, but let's just consider an example:

$$\begin{tikzcd} \mathrm{Grp} \arrow[d, shift left, "R"] \\ \mathrm{Set} \arrow[u, shift left, "L"] \end{tikzcd}$$

using our parentheses notation where, for example,

$$(g)(h) \qquad\qquad g, h \in G$$

means the product in LRG, which is different from

$$(gh).$$

Let $S \in \mathrm{Set}$ & look at the zig-zag:

$$\begin{tikzcd} LS \arrow[d, "L(\eta_S)"] & s_1 s_2 \cdots s_n \arrow[d, mapsto] & s_i \in S \\ LRLS \arrow[d, "\varepsilon_{LS}"] & (s_1 s_2 \cdots s_n) \arrow[d, mapsto] & \\ LS & s_1 s_2 \cdots s_n & \end{tikzcd}$$

$\eta_S : S \to RLS$

$s \mapsto s$

so $L(\eta_S) : LS \to LRLS$

$s_1 \cdots s_n \mapsto (s_1 \cdots s_n)$

$\varepsilon_G : LRG \to G \qquad G \in \mathrm{Grp}$

$(g_1)\cdots(g_n) \mapsto g_1 \cdots g_n$

so $\varepsilon_{LS} : LRLS \to LS$

$(s_{11}\cdots)\cdots(s_{1n}\cdots) \mapsto s_{11}\cdots s_{1n}\cdots$

So the zig-zag identity holds.

We could also prove  = , but let's again just do an example: consider

$$\begin{tikzcd} \mathrm{Grp} \arrow[d, shift left, "R"] \\ \mathrm{Set} \arrow[u, shift left, "L"] \end{tikzcd}$$

& take $G \in \mathrm{Grp}$.

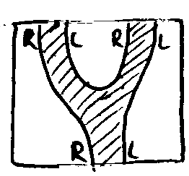
$$\begin{tikzcd} RG \arrow[d, "\eta_{RG}"] & g \arrow[d, mapsto] \\ RLRG \arrow[d, "R(\varepsilon_G)"] & (g) \arrow[d, mapsto] \\ RG & g \end{tikzcd}$$

$g \in G$

$\varepsilon_G : LRG \to G$
$(g_1)\cdots(g_n) \mapsto g_1 \cdots g_n$

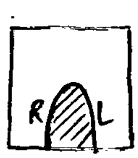
$R(\varepsilon_G) : RLRG \to RG$
$(g_1)\cdots(g_n) \mapsto g_1 \cdots g_n$

Next, using the zig-zag identities, we can build a monoid, with product:



$$RLRL \xrightarrow{R\varepsilon L} RL$$

and unit



$$1_C \xrightarrow{\eta} RL$$

Note RL is an object in $\mathrm{End}(\mathcal{C})$, the category with

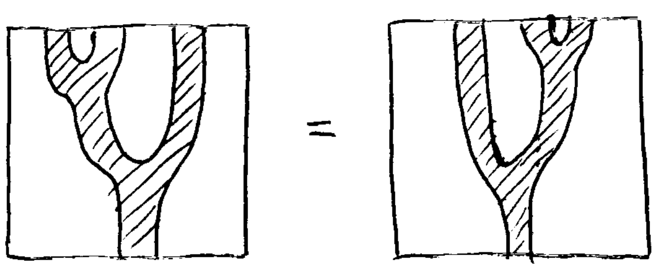
- functors $\mathcal{C} \xrightarrow{F} \mathcal{C}$ as objects
- natural transformations $\begin{tikzcd} \mathcal{C} \arrow[r, bend left, "F"] \arrow[r, bend right, "G"'] \arrow[r, phantom, "\Downarrow\alpha"] & \mathcal{C} \end{tikzcd}$ as morphisms.

$\mathrm{End}(\mathcal{C})$ is a monoidal category with tensor product given by composition

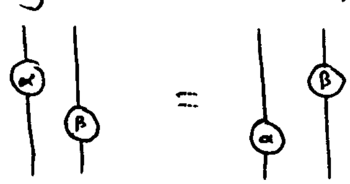
$$\otimes : \mathrm{End}(\mathcal{C}) \times \mathrm{End}(\mathcal{C}) \longrightarrow \mathrm{End}(\mathcal{C})$$
$$(F, G) \longmapsto FG$$

which in our diagram notation is just drawn as "horizontal juxtaposition."

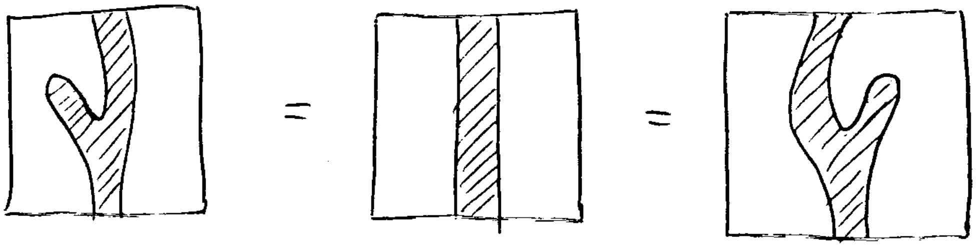
Why is RL a monoid? Need associativity:



this follows from the interchange law – true in any monoidal category (in fact, any 2-category):



We also need l/r unit laws:



These are easy to prove from the zig-zag identities.

So — any adjunction $\begin{array}{c} D \\ {\scriptstyle L}\uparrow\downarrow{\scriptstyle R} \\ C \end{array}$ gives a monoid in End(C).

A monoid in an endofunctor category, e.g. End(C), is called a <u>monad</u>.

Since Δ_{alg} is the <u>free</u> monoidal category on a monoid, we thus get a monoidal functor

$$\Delta_{alg} \longrightarrow \mathrm{End}(C)$$

$$1 \longmapsto LR$$

$$1+1 \to 1 \longmapsto L\varepsilon R : LRLR \Rightarrow LR$$

$$0 \to 1 \longmapsto \iota : 1_C \Rightarrow LR$$

But we wanted simplicial objects from adjunctions. A simplicial object in X is a functor

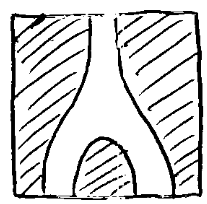
$$\Delta_{alg}^{op} \longrightarrow X$$

So, we're getting a simplicial object

$$\Delta_{alg}^{op} \longrightarrow End(C)^{op}.$$

But better: don't use the monoid object in $End(C)$, use the comonoid object in $End(D)$! This has

coproduct



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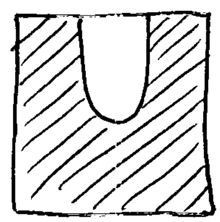

This will give a functor

$$\Delta_{alg}^{op} \longrightarrow End(D)$$

—i.e. a simplicial object. This gives the bar construction