

Recall: given smooth categories C & D , a smooth anafunctor $F: C \rightarrow D$ consists of:

1) An open cover U_i of $Ob(C)$

2) Smooth functors

$$F_i: C|_{U_i} \rightarrow D$$

3) Smooth natural isomorphisms

$$\begin{array}{ccc}
 & F_j & \\
 & \curvearrowright & \\
 C|_{U_{ij}} & \Downarrow g_{ij} & \rightarrow D \\
 & \curvearrowleft & \\
 & F_i &
 \end{array}$$

such that

4) the cocycle condition

$$g_{ij}g_{jk} = g_{ik}$$

holds on $C|_{U_{ijk}}$.

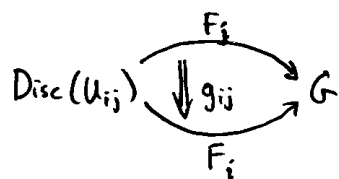
So: what's a smooth anafunctor

$$F: Disc(M) \rightarrow G ?$$

1) An open cover U_i of $Ob(Disc(M)) = M$

2) smooth functors $F_i: Disc(U_i) \rightarrow G$ — but there's only one, since $Disc(U_i)$ is discrete & G has just one object.

3) Smooth natural isomorphisms



For each object in $\text{Disc}(U_{ij})$, i.e. $x \in U_{ij}$, this gives an isomorphism $g_{ij}(x) : F_j(x) \rightarrow F_i(x)$ where $* \in G$ is unique.

In short, $g_{ij} \in G$. Why is g_{ij} natural?

$$\begin{array}{ccc} F_j(x) & \xrightarrow{F_j(1_x)} & F_j(x) \\ g_{ij}(x) \downarrow & & \downarrow g_{ij}(x) \\ F_i(x) & \xrightarrow{F_i(1_x)} & F_i(x) \end{array}$$

must commute, and obviously does!

4) And we require $g_{ij}g_{jk} = g_{ik}$

All this is just a Čech 1-cocycle! So:

$$\{\text{smooth anafunctors } F: \text{Disc}(M) \rightarrow G\} \cong \{\check{C}\text{ech } 1\text{-cocycles}\}$$

so we can cook up a definition of "anatural transformation" s.t.

$$\frac{\{\text{smooth anafunctor } F: \text{Disc}(M) \rightarrow G\}}{\text{anatural isomorphism}} \cong \check{H}^1(M, G)$$

In general:

Def: Given smooth anafunctors $C \begin{array}{c} \xrightarrow{F} \\ \xrightarrow{F'} \end{array} D$

a smooth ananatural transformation $C \begin{array}{c} \xrightarrow{F} \\ \Downarrow f \\ \xrightarrow{F'} \end{array} D$ is

a collection of smooth natural transformations

$$C|_{U_i} \begin{array}{c} \xrightarrow{F_i} \\ \Downarrow f_i \\ \xrightarrow{F'_i} \end{array} D$$

such that

$$f_i g_{ij} = g'_{ij} f_j \quad \forall i, j$$

where g_{ij}, g'_{ij} come from the anafunctors F, F' in the way we described.

In summary:

$$\frac{\{G\text{-bundles over } M\}}{\text{isomorphism}} \cong \check{H}^1(M, G) \cong \frac{\{\text{smooth anafunctors } F: \text{Disc}(M) \rightarrow G\}}{\text{ananatural isomorphism}}$$

but in fact:

$$\frac{\{G \text{ bundles with connection over } M\}}{\text{isomorphism}} \cong \frac{\{\text{smooth anafunctors } \{ \text{hol}: PM \rightarrow G \}\}}{\text{ananatural isomorphism}}$$

this deserves to be called " $\check{H}^1(PM, G)$ " — now we're

really doing Čech cohomology of a smooth category!

More generally given two smooth 1-categories C & D ,
let's define

$$\check{H}^1(C, D) = \frac{\{\text{smooth anafunctors } F: C \rightarrow D\}}{\text{anatural isomorphism}}$$

Then replace 1 by "n"!