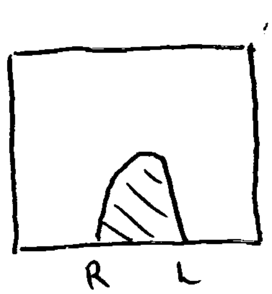
The Bar Construction

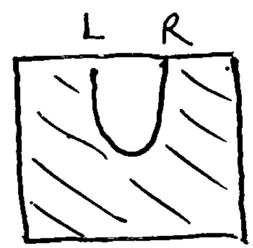
Suppose we have an adjunction:

$$\begin{tikzcd} D \arrow[d, shift left, "R"] \\ C \arrow[u, shift left, "L"] \end{tikzcd}$$

We get a unit & counit:



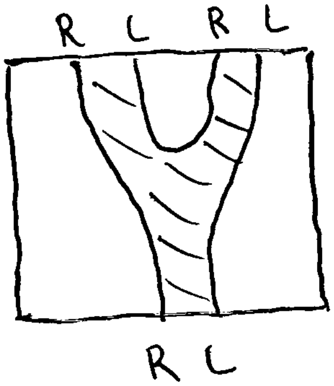
unit $\iota : 1_C \Rightarrow RL$



counit $\varepsilon : LR \Rightarrow 1_D$

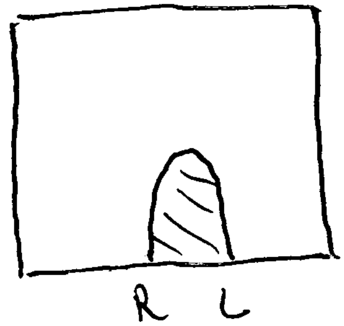
satisfying zig-zag identities. Last time we saw this gives a <u>monad on C</u>, i.e. a monoid in $End(C)$, namely $RL \in End(C)$.

This is indeed a monoid:



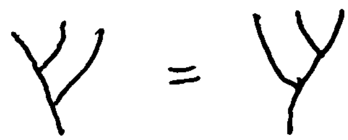
multiplication

$RLRL \xRightarrow{R\varepsilon L} RL$



unit

$1_C \xRightarrow{\eta} RL$

satisfying associativity: 

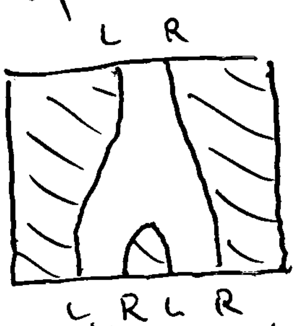
& l/r unit laws: 

We also get a <u>comonad on D</u>, i.e.

a comonoid in End(D), namely $LR \in$ End(D).

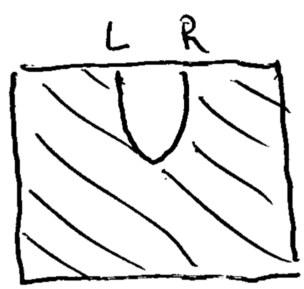
A "comonoid" is just like a monoid,

but upside down:



comultiplication

$LR \xRightarrow{L\eta R} LRLR$



counit

$LR \xRightarrow{\varepsilon} 1_D$

These satisfy coassociativity: 

& l/r counit laws: 

More tersely, if M is a monoidal category, M^{op} is also monoidal with same $\otimes$, and:

Def – A <u>comonoid</u> in M is a monoid in M^{op}.

E.g. – An <u>algebra</u> is a monoid in Vect, a <u>coalgebra</u> is a comonoid in Vect, or monoid in Vect^{op}.

So: our adjunction gives a comonad LR, which is a monoid in $\mathrm{End}(D)^{op}$. Since Δ is the free monoidal category on a monoid, this gives a monoidal functor

$$\Delta \longrightarrow \mathrm{End}(D)^{op}$$

i.e. a monoidal functor

$$\Delta^{op} \xrightarrow{\alpha} \mathrm{End}(D)$$

and thus a simplicial object in $\mathrm{End}(D)$!

Taking a specific object $d \in D$, we get:

$$\begin{aligned} ev_d : \mathrm{End}(D) &\longrightarrow D \\ F &\longmapsto Fd \end{aligned}$$

so we get a simplicial object in D:

$$\Delta^{op} \xrightarrow{\alpha} \mathrm{End}(D) \xrightarrow{ev_d} D$$

This simplicial object in D is called $\underline{d} : \Delta^{op} \to D$; we call this the <u>bar construction</u>.

Moral: given an adjunction $\begin{array}{c} D \\ L \uparrow \downarrow R \\ C \end{array}$, any object $d \in D$ gives a simplicial object $\underline{d}$ in D.

Example: The Cohomology of Groups

Here we take a group G & get an adjunction

$$\begin{tikzcd} G\text{-Set} \arrow[d, shift left, "R"] \\ \text{Set} \arrow[u, shift left, "L"] \end{tikzcd}$$

where G-Set is the category of sets w. left G-action.

So, given a G-Set X, we get:

(-1)-simplices   0-simplicies   1-simplicies

$$\underline{X} = \begin{tikzcd} X & LRX \arrow[l, "\varepsilon_X"'] \arrow[r, bend left, "L(\eta_{RX})"] & LRLRX \arrow[l, shift right, "\varepsilon_{LRX}"'] \arrow[l, bend left=50, "LR(\varepsilon_X)"] \arrow[r, shift left] \arrow[r, shift left=3] & \cdots \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \end{tikzcd}$$

a simplicial G-set!

What's a 1-simplex, or 2-simplex, in this simplicial G-set like?

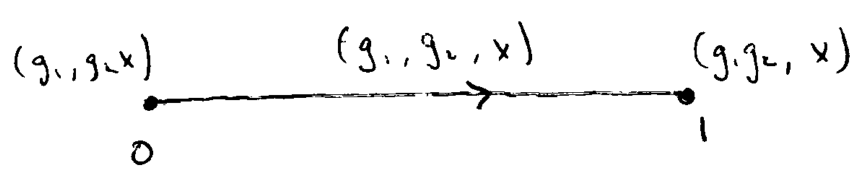
Given our G-set X, what's LRX?
RX is usually just called "X" - the underlying set of our G-set X. LRX has elements "gx" for each $g \in G$ & $x \in X$. Really "gx" is just (g, x), so $LRX = G \times X$, which is a G-set with

$$g_1 (g_2, x) = (g_1 g_2, x).$$

So

$$
\begin{array}{ccccccc}
 & & \text{(-1)-simplices} & & \text{0-simplices} & & \text{1-simplices} \\
\underline{X} & = & X & \xleftarrow{\varepsilon_X} & G \times X & \xleftarrow{\varepsilon_{LRX}} & G \times G \times X \\
 & & gx & \longleftarrow & (g, x) \quad (g_1 g_2, x) & \longleftarrow & (g_1, g_2, x) \\
 & & & & & \xleftarrow{LR(\varepsilon_X)} & \\
 & & & & (g_1, g_2 x) & \longleftarrow & (g_1, g_2, x)
\end{array}
$$

So, a typical 1-simplex in $\underline{X}$ looks like:



Note: both 0-simplices here have as face the (-1)-simplex g_1g_2x. So this 1-simplex is a proof that $g_1(g_2x) = (g_1g_2)x$ — the 2 formal expressions (g_1, g_2x) & (g_1g_2, x) evaluate via ε_X to the same element of X, namely g_1g_2x.

How about a 2-simplex in $\underline{X}$?

$$\begin{tikzcd}
 & {1 \quad (g_1g_2,\ g_3x)} \arrow[dr, "{(g_1g_2,\ g_3,\ x)}"] & \\
{(g_1,g_2g_3x) \quad 0} \arrow[ur, "{(g_1,\ g_2,\ g_3x)}"] \arrow[rr, "{(g_1,\ g_2g_3,\ x)}"'] \arrow[rr, phantom, shift left=4ex, "{(g_1,g_2,g_3,x)}"] & & {2 \quad (g_1g_2g_3,\ x)}
\end{tikzcd}$$

Here we see 2 proofs that $g_1(g_2g_3x) = (g_1g_2g_3)x$:

$$\begin{tikzcd}
 & (g_1g_2)(g_3x) \arrow[dr, "(g_1g_2)g_3x"] & \\
g_1(g_2g_3x) \arrow[ur, "g_1g_2(g_3x)"] \arrow[rr, "g_1(g_2g_3)x"'] & & (g_1g_2g_3)x
\end{tikzcd}$$

using one step or 2. The triangle is a "metaproof" or "syzygy" – a "homotopy between proofs".

The simplicial G-set $\underline{X}$ is called EG when $X = *$. In general $\underline{X}$ has contractible components, one for each element of X – these are the 1-simplicies of $\underline{X}$!
So EG has one contractible component.
It's like a "puffed-up point" – a contractible space on which G acts freely.