5 June 2007

Quantization & Cohomology

Review & Prospectus

We've been trying to understand classical and quantum physics in a unified way, starting with a category C of "configurations" (ways things can be) and "processes" (ways things can become), with a functor

$$S : C \longrightarrow \mathbb{R}$$

or

$$\Phi : C \longrightarrow U(1)$$

assigning to any process $\gamma : x \longrightarrow y$ its "action" $S(\gamma) \in \mathbb{R}$ or "phase" $\Phi(\gamma) \in U(1)$. These are often related by

$$\Phi(\gamma) = e^{iS(\gamma)}$$

but Φ needs not be of this form.

CLASSICAL	QUANTUM
C smooth category;	C measurable category
$\Phi : C \to U(1)$ smooth functor	$\Phi : C \to U(1)$ mble. functor
Given $x, y \in C$ find $\gamma : x \to y$ with $\delta \Phi(\gamma) = 0$	Given $x, y \in C$ find $\langle y, x\rangle = \int_{\gamma : x \to y} \Phi(\gamma) \mathcal{D}\gamma \in \mathbb{C}$

We can try to unify these using <u>rigs</u>, but in both cases there are subtleties that complicate the picture above. In the quantum case, we need to stretch the theory of integration beyond traditional measure theory to do the "path integral" above.

In the classical case, smooth functors aren't enough — we need smooth anafunctors. So, there's a lot more to be understood, both classically & quantumly.

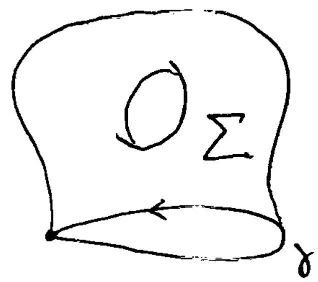
Look at the classical side of the prospectus. In classical mechanics we often start with a phase space (X, ω) — a symplectic manifold. From this we build a smooth category

$$C = PX$$

the path groupoid. Then we seek a smooth anafunctor:

$$\Phi : PX \longrightarrow U(1)$$

such that if γ is a loop that bounds a surface Σ:



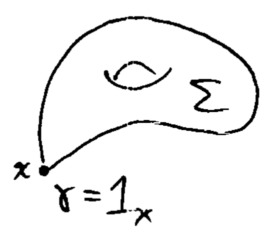
$$\partial \Sigma = \gamma$$

we have

$$\Phi(\gamma) = e^{i\int_\Sigma \varpi}$$

There could be many Φ's, or no Φ's, that do this.
For such a Φ to exist, we need

$$\partial \Sigma = \emptyset \implies \int_\Sigma \varpi \in 2\pi\mathbb{Z}$$



$$\partial \Sigma = \emptyset$$

Since we need $\Phi(1_x) = 1$. We say a closed 2-form is integral if $\int_\Sigma \varpi \in 2\pi\mathbb{Z}$ for all closed $\Sigma \subseteq X$.

Conversely, if ϖ is integral, we can indeed find Φ as desired. If ϖ is integral

$$
\begin{array}{c}
[\varpi] \in H^2(X,\mathbb{Z}) \\
\cong \\
\check{H}^1(X,U(1))
\end{array}
$$

and we've seen $\check{H}^1(X,U(1))$ classifies principal $U(1)$-bundles over X. In fact, our smooth anafunctor

$$\Phi : PX \longrightarrow U(1)$$

arises as follows: pick a $U(1)$ bundle $\begin{array}{c} P \\ \downarrow \\ X \end{array}$ corresponding to $[\varpi]$, pick a connection A on it, & let

$$\Phi(\gamma) = \mathrm{hol}(\gamma)$$

where holonomy is defined using A.

To go further, we should use geometric quantization to get a Hilbert space from $(X, \varpi, \begin{array}{c} P \\ \downarrow \\ X \end{array}, A)$ — this requires extra structure on X, e.g. a Kähler structure. It would be great to show that these Hilbert spaces match those given by our path integral procedure!

Also, in this course we <u>hinted</u> at how to categorify all this stuff, to go from particle physics to string physics. Naively, we could categorify our previous chart:

CLASSICAL

C smooth 2-category,

$$\Phi : C \longrightarrow U(1)\mathrm{Tor} \cong U(1)[1]$$

a smooth 2-functor. Given

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_2"'] & y \end{tikzcd}$$

find $\Sigma : \gamma_1 \Rightarrow \gamma_2$ s.t.

$$\delta \Phi(\Sigma) = 0$$

QUANTUM

C measurable 2-category

$$\Phi : C \longrightarrow U(1)[1]$$

a measurable 2-functor. Given

$$\begin{tikzcd} x \arrow[r, bend left, "\gamma_1"] \arrow[r, bend right, "\gamma_2"'] & y \end{tikzcd}$$

we can compute an amplitude

$$\langle \gamma_2, \gamma_1 \rangle = \int_{\Sigma : \gamma_1 \Rightarrow \gamma_2} \Phi(\Sigma)\, \mathcal{D}\Sigma$$

The path integrals become a lot harder now, but still manageable. Over on the classical side, we really need 2-anafunctors.

Urs Schreiber & J.B. showed:

Given a smooth space X, there's a smooth 2-groupoid P_2X where:

- objects are points of X
- morphisms are smooth paths in X
- 2-morphisms are thin homotopy classes of paths-of-paths in X

Given a "smooth 2-group" G (e.g. $U(1)[1]$), 2-connections on principal G-2-bundles over X correspond to smooth 2-anafunctors

$$\mathrm{hol}\colon P_2X \longrightarrow G$$

So – everything we said about bundles & connections categorifies!

In string theory there are nice examples of smooth 2-groups. For example, given any compact simple (simply connected) Lie group H, there's a 2-group "String_H". When $H = \mathrm{Spin}(n)$ is the double cover of $SO(n)$, "String_H" is the symmetry 2-group involved in studying "spinning strings", and basic in elliptic cohomology, as studied by Stolz & Teichner.