Cohomology of Algebraic Gadgets

Given an adjunction $\begin{array}{c} D \\ L\uparrow\downarrow R \\ C \end{array}$ & an object $d \in D$, the bar construction gives a simplicial object

$$\bar{d} : \Delta^{op}_{alg} \longrightarrow D$$

which looks like:

$$\begin{tikzcd}
d & LRd \arrow[l, "\varepsilon_d"'] \arrow[r, bend left] & (LR)^2 d \arrow[l, shift left] \arrow[l, shift right] \arrow[r, shift left, bend left] \arrow[r, shift right, bend left] & (LR)^3 d \arrow[l] \arrow[l, shift left=2] \arrow[l, shift right=2] \\
-1\text{-simplices} & 0\text{-simplices} & 1\text{ simplices} & 2\text{-simplices}
\end{tikzcd}$$

We could also ignore the (-1)-simplices and get

$$\bar{d} : \Delta^{op}_{top} \longrightarrow D$$

but the (-1)-simplices are important! Let's assume there's a (faithful) forgetful functor $U : D \longrightarrow \mathrm{Set}$. Then an element of Ud is a -1-simplex, and our topologist's simplicial set $U\bar{d} : \Delta^{op}_{top} \longrightarrow \mathrm{Set}$ has "components", one for each (-1)-simplex. 0-simplices are "formal expressions" (elements of $ULRd$) & 2 formal expressions evaluating via ε_d to the same element of Ud lie in the same "component".

In Todd Trimble's notes, there's a proof that, under mild conditions, each component is contractible.

So $\bar{d}$ is a "puffed up" version of d in which equations have been replaced by edges, etc... We can use this to study "holes" in algebraic gadgets. The classic case is when our algebraic gadgets are R-modules for some ring R:

$$\begin{array}{c} D = RMod \\ \uparrow\downarrow \\ C = AbGp \end{array}$$

This covers a lot of ground:

- "Ext^i" and "Tor_i" — invariants of an R-module M.
- "group cohomology" "group homology" — invariants of a group G obtained via the group ring $R = \mathbb{Z}[G]$

- "Lie algebra cohomology"

  "Lie algebra homology"

  – invariants of a Lie algebra $\mathfrak{g}$, obtained via the universal enveloping ring $R = U\mathfrak{g}$

How do these work? The basic examples are Ext & Tor, so let's do those. Suppose R is a ring & $M \in R\text{Mod}$. Then via

$$\begin{array}{c} R\text{Mod} \\ \uparrow\downarrow \\ \text{AbGp} \end{array}$$

the bar construction gives a simplicial R-module $\overline{M}$. This has an underlying simplicial abelian group, i.e. a chain complex of abelian groups. But what a simplicial R-module is, is a chain complex of R-modules:

$$\overline{M} = M \leftarrow M_0 \leftarrow M_1 \leftarrow M_2 \leftarrow \cdots$$

where the (-1)-chains are elts of our original R-module, M.

In fact, we have:

1) $H_i(\overline{M}) = \begin{cases} M & i=-1 \\ 0 & i \geq 0 \end{cases}$

(an algebraic analogue of how the simplicial set $U\bar{d}$ mentioned above consists of contractible components, one for each elt of d)

2) All M_i are free for $i \geq 0$

We summarize 1) & 2) by saying $\overline{M}$ is a free resolution of M. Given any other R-module, say A, we study M by homming $\overline{M}$ into A. Form the cochain complex of R-modules:

$$\hom(\overline{M}, A) = \left\{ \hom(M,A) \longrightarrow \hom(M_0, A) \longrightarrow \hom(M_1, A) \longrightarrow \cdots \right\}$$

and take its cohomology to get "Ext":

$$\mathrm{Ext}^i_R(M,A) := H^i(\hom(\overline{M}, A))$$

This detects "A-valued holes in M".

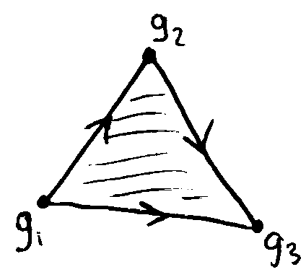
From this we get group cohomology:

$$H^i(G,A) = \mathrm{Ext}^i_R(\mathbb{Z},A)$$

where $R=\mathbb{Z}[G]$, $\mathbb{Z}$ is the trivial R-module, and A is any R-module, i.e. an abelian group on which G acts. We can also compute this in an explicitly topological way: take

$$\begin{tikzcd} G\text{-Set} \arrow[d, shift left] \\ \text{Set} \arrow[u, shift left] \end{tikzcd}$$

& do the bar construction to the terminal G-set, the point $*$, getting the space EG:
(a simplicial G-set)



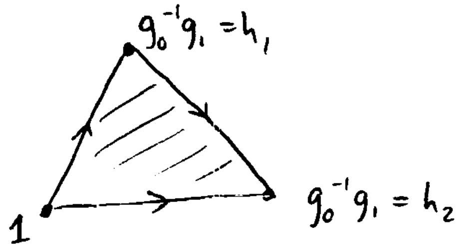
a typical triangle in EG

If A is just an abelian group with trivial G-action,

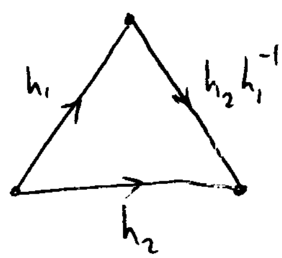
$$H^i(G,A) \cong H^i(EG/G, A)$$

where $H^i(EG/G, A)$ is the cohomology of the space $EG/G = BG$, the classifying space of G.

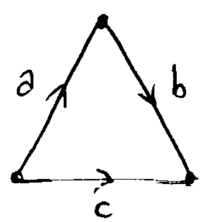
A typical triangle in BG looks like this:



or



i.e.



w. $ab = c$
$a, b, c \in G$

– the <u>nerve</u> of the group G regarded as a category!

Going back to our R-module M & its free resolution

$$\overline{M} = M \leftarrow M_0 \leftarrow M_1 \leftarrow \cdots$$

we could also tensor this with any R-module A getting:

$$A \otimes \overline{M} = \{ A \otimes M \leftarrow A \otimes M_0 \leftarrow A \otimes M_1 \leftarrow \cdots \}$$

– a chain complex of R-modules. This gives us "Tor":

$$\mathrm{Tor}_i(M, A) = H_i(A \otimes \overline{M})$$