Some Definitions Everyone Should Know

John C. Baez, May 13, 2010

A topological quantum field theory is a 'symmetric monoidal functor' $Z\colon n\mathrm{Cob} \to \mathrm{Vect}$. To know what this means, we need some definitions from category theory. These make for rather dry reading unless you get into the right mood, but they're worth knowing, since they show up all over the place.

Categories, Functors, Natural Transformations

Definition 1. *A* **category** C *consists of:*

- *a collection* $\mathrm{Ob}(C)$ *of* **objects**.
- *for any pair of objects* x, y*, a set* $\hom(x,y)$ *of* **morphisms** *from* x *to* y*. (If* $f \in \hom(x,y)$ *we write* $f\colon x \to y$*.)*

equipped with:

- *for any object* x*, an* **identity morphism** $1_x\colon x \to x$*.*
- *for any pair of morphisms* $f\colon x \to y$ *and* $g\colon y \to z$*, a morphism* $fg\colon x \to z$ *called the* **composite** *of* f *and* g*.*

such that:

- *for any morphism* $f\colon x \to y$*, the* **left and right unit laws** *hold:* $1_x f = f = f 1_y$*.*
- *for any triple of morphisms* $f\colon w \to x$*,* $g\colon x \to y$*,* $h\colon y \to z$*, the* **associative law** *holds:* $(fg)h = f(gh)$*.*

We usually write $x \in C$ as an abbreviation for $x \in \mathrm{Ob}(C)$. An **isomorphism** is a morphism $f\colon x \to y$ with an **inverse**, i.e. a morphism $g\colon y \to x$ such that $fg = 1_x$ and $gf = 1_y$.

Definition 2. *Given categories* C, D*, a* **functor** $F\colon C \to D$ *consists of:*

- *a function* $F\colon \mathrm{Ob}(C) \to \mathrm{Ob}(D)$*.*
- *for any pair of objects* $x, y \in \mathrm{Ob}(C)$*, a function* $F\colon \hom(x,y) \to \hom(F(x), F(y))$*.*

such that:

- F **preserves identities***: for any object* $x \in C$*,* $F(1_x) = 1_{F(x)}$*.*
- F **preserves composition***: for any pair of morphisms* $f\colon x \to y$*,* $g\colon y \to z$ *in* C*,* $F(fg) = F(f)F(g)$*.*

It's not hard to define identity functors and composition of functors, and to check the left and right unit law and associative law for these.

Definition 3. *Given functors* $F, G\colon C \to D$*, a* **natural transformation** $\alpha\colon F \Rightarrow G$ *consists of:*

- *a function* α *mapping each object* $x \in C$ *to a morphism* $\alpha_x\colon F(x) \to G(x)$

such that:

- *for any morphism $f: x \to y$ in C, this diagram commutes:*

$$\begin{array}{ccc} F(x) & \xrightarrow{F(f)} & F(y) \\ {\scriptstyle \alpha_x}\downarrow & & \downarrow{\scriptstyle \alpha_y} \\ G(x) & \xrightarrow[G(f)]{} & G(y) \end{array}$$

With a little thought you can figure out how to compose natural transformations $\alpha: F \Rightarrow G$ and $\beta: G \Rightarrow H$ and get a natural transformation $\alpha\beta: F \Rightarrow H$. We can also define identity natural transformations. Again, it's not hard to check the left and right unit law and associativity for these.

Definition 4. *Given functors $F, G: C \to D$, a* **natural isomorphism** *$\alpha: F \Rightarrow G$ is a natural transformation that has an* **inverse***, i.e. a natural transformation $\beta: G \Rightarrow F$ such that $\alpha\beta = 1_F$ and $\beta\alpha = 1_G$.*

It's not hard to see that a natural transformation $\alpha: F \Rightarrow G$ is a natural isomorphism iff for every object $x \in C$, the morphism α_x is invertible.

Definition 5. *A functor $F: C \to D$ is an* **equivalence** *if it has a* **weak inverse***, that is, a functor $G: D \to C$ such that there exist natural isomorphisms $\alpha: FG \Rightarrow 1_C$, $\beta: GF \Rightarrow 1_D$.*

Monoidal, Braided Monoidal, and Symmetric Monoidal Categories

Definition 6. *A* **monoidal category** *consists of:*

- *a category M.*
- *a functor called the* **tensor product** *$\otimes: M \times M \to M$, where we write $\otimes(x, y) = x \otimes y$ and $\otimes(f, g) = f \otimes g$ for objects $x, y \in M$ and morphisms f, g in M.*
- *an object called the* **identity object** *$1 \in M$.*
- *natural isomorphisms called the* **associator***:*

$$a_{x,y,z}: (x \otimes y) \otimes z \to x \otimes (y \otimes z),$$

the **left unit law***:*

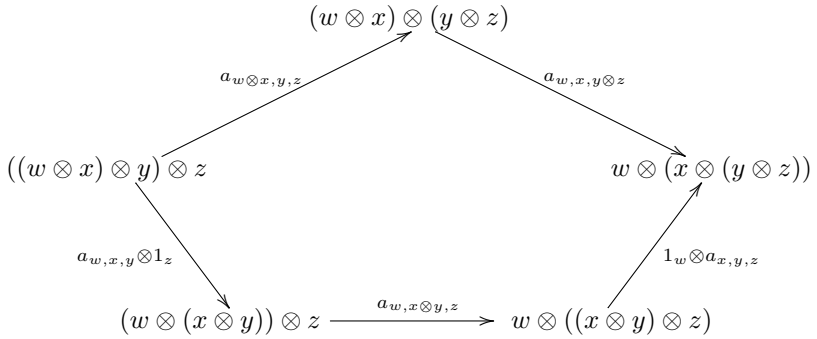
$$\ell_x: 1 \otimes x \to x,$$

and the **right unit law***:*

$$r_x: x \otimes 1 \to x.$$

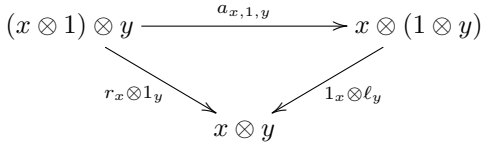
such that the following diagrams commute for all objects $w, x, y, z \in M$:

- *the* **pentagon equation***:*



governing the associator.

- *the* **triangle equations***:*



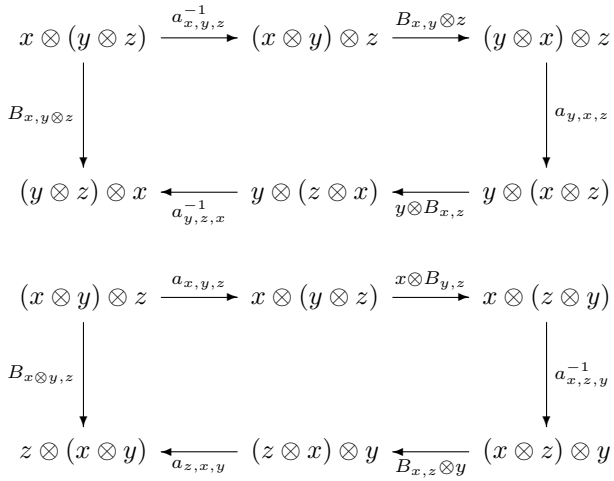
governing the left and right unit laws.

Definition 7. *A* **braided monoidal category** *consists of:*

- *a monoidal category M.*
- *a natural isomorphism called the* **braiding***:*

$$B_{x,y} \colon x \otimes y \to y \otimes x.$$

such that these two diagrams commute, called the **hexagon equations***:*



Definition 8. *A* **symmetric monoidal category** *is a braided monoidal category M for which the braiding satisfies $B_{x,y} = B^{-1}_{y,x}$ for all objects x and y.*

A monoidal, braided monoidal, or symmetric monoidal category is called **strict** if $a_{x,y,z}, \ell_x, r_x$ are all identity morphisms. In this case we have

$$(x \otimes y) \otimes z = x \otimes (y \otimes z),$$

$$1 \otimes x = x, \qquad x \otimes 1 = x.$$

In a sense to be made precise below, Mac Lane has shown that every monoidal (resp. braided monoidal, symmetric monoidal) category is equivalent to a strict one. However, the examples that turn up 'in nature', like the category of vector spaces and linear maps equipped with its usual tensor product, are rarely strict.

Monoidal, Braided Monoidal, and Symmetric Monoidal Functors

We can ask that a functor between monoidal categories preserve the tensor product and the identity object. This gives the notion of a 'monoidal functor'. However, we should only ask that it preserves these things *up to a specified isomorphism*. This isomorphism should then satisfy some compatibility conditions.

Definition 9. *A functor $F\colon C \to C'$ between monoidal categories is* **monoidal** *if it is equipped with:*

- *a natural isomorphism $\Phi_{x,y}\colon F(x) \otimes F(y) \to F(x \otimes y)$.*
- *an isomorphism $\phi\colon 1_{C'} \to F(1_C)$.*

such that

- *the following diagram commutes for any objects $x, y, z \in C$:*

$$\begin{array}{ccccc}
(F(x)\otimes F(y))\otimes F(z) & \xrightarrow{\Phi_{x,y}\otimes 1_{F(z)}} & F(x\otimes y)\otimes F(z) & \xrightarrow{\Phi_{x\otimes y,z}} & F((x\otimes y)\otimes z) \\
\downarrow{\scriptstyle a_{F(x),F(y),F(z)}} & & & & \downarrow{\scriptstyle F(a_{x,y,z})} \\
F(x)\otimes(F(y)\otimes F(z)) & \xrightarrow{1_{F(x)}\otimes\Phi_{y,z}} & F(x)\otimes F(y\otimes z) & \xrightarrow{\Phi_{x,y\otimes z}} & F(x\otimes(y\otimes z))
\end{array}$$

- *the following diagrams commute for any object $x \in C$:*

$$\begin{array}{ccc}
1\otimes F(x) & \xrightarrow{\ell_{F(x)}} & F(x) \\
\downarrow{\scriptstyle \phi\otimes 1_{F(x)}} & & \uparrow{\scriptstyle F(\ell_x)} \\
F(1)\otimes F(x) & \xrightarrow{\Phi_{1,x}} & F(1\otimes x)
\end{array}$$

$$\begin{array}{ccc}
F(x)\otimes 1 & \xrightarrow{r_{F(x)}} & F(x) \\
\downarrow{\scriptstyle 1_{F(x)}\otimes\phi} & & \uparrow{\scriptstyle F(r_x)} \\
F(x)\otimes F(1) & \xrightarrow{\Phi_{x,1}} & F(x\otimes 1)
\end{array}$$

We similarly have concepts of 'braided monoidal functor' and 'symmetric monoidal functor':

Definition 10. *A functor $F\colon C \to C'$ between braided monoidal categories is* **braided monoidal** *if it is monoidal and it makes the following diagram commute for all $x, y \in C$:*

$$\begin{array}{ccc}
F(x)\otimes F(y) & \xrightarrow{B_{F(x),F(y)}} & F(y)\otimes F(x) \\
\downarrow{\scriptstyle \Phi_{x,y}} & & \downarrow{\scriptstyle \Phi_{y,x}} \\
F(x\otimes y) & \xrightarrow{F(B_{x,y})} & F(y\otimes x)
\end{array}$$

A **symmetric monoidal functor** is simply a braided monoidal functor that happens to go between symmetric monoidal categories! No extra condition is involved here.

Monoidal, Braided Monoidal, and Symmetric Monoidal Natural Transformations

It would be a pity to discuss monoidal, braided monoidal and symmetric monoidal *categories* and *functors* but not the corresponding sorts of *natural transformations*. Recall that a monoidal functor $F\colon C \to C'$ is really a triple (F, Φ, ϕ) consisting of a functor, a natural isomorphism

$$\Phi_{x,y}\colon F(x) \otimes F(y) \to F(x \otimes y),$$

and an isomorphism

$$\phi\colon 1_{C'} \to F(1_C).$$

A 'monoidal natural transformation' is one that gets along with these extra isomorphisms:

Definition 11. *Suppose that (F, Φ, ϕ) and (G, Γ, γ) are monoidal functors from the monoidal category C to the monoidal category D. Then a natural transformation $\alpha\colon F \to G$ is* **monoidal** *if the diagrams*

$$\begin{array}{ccc} F(x) \otimes F(y) & \xrightarrow{\alpha_x \otimes \alpha_y} & G(x) \otimes G(y) \\ \downarrow{\scriptstyle \Phi_{x,y}} & & \downarrow{\scriptstyle \Gamma_{x,y}} \\ F(x \otimes y) & \xrightarrow{\alpha_{x \otimes y}} & G(x \otimes y) \end{array}$$

and

$$\begin{array}{ccc} 1 & & \\ \downarrow{\scriptstyle \phi} & \searrow{\scriptstyle \gamma} & \\ F(1) & \xrightarrow{\alpha_1} & G(1) \end{array}$$

commute.

There are no extra conditions required of **braided monoidal** or **symmetric monoidal** natural transformations.

One needs these concepts to give a precise statement of the sense in which any monoidal (resp. braided monoidal, symmetric monoidal) category is equivalent to a strict one, since the right concept of 'equivalence' is stronger than mere equivalence as categories:

Definition 12. *If C and D are monoidal categories, a monoidal functor $F\colon C \to D$ is an* **monoidal equivalence** *if there is a monoidal functor $G\colon D \to C$ such that there exist monoidal natural isomorphisms $\alpha\colon GF \Rightarrow 1_C$, $\beta\colon FG \Rightarrow 1_D$.*

And similarly:

Definition 13. *If C and D are braided (resp. symmetric) monoidal categories, a braided (resp. symmetric) monoidal functor $F\colon C \to D$ is an* **braided (resp. symmetric) monoidal equivalence** *if there is a braided (resp. symmetric) monoidal functor $G\colon D \to C$ such that there exist braided (resp. symmetric) monoidal natural isomorphisms $\alpha\colon GF \Rightarrow 1_C$, $\beta\colon FG \Rightarrow 1_D$.*

Theorem 1 – MacLane's Theorem. *Given a monoidal category C, there exists a strict monoidal category C' for which there is a monoidal equivalence $F\colon C \to C'$. Similarly, given a braided (resp. symmetric) monoidal category, there exists a strict braided (resp. symmetric) monoidal category C' for which there is a braided (resp. symmetric) monoidal equivalence $F\colon C \to C'$.*

With some work one can check that there is a 2-category **Cat** consisting of categories, functors and natural transformations. Similarly there is a 2-category **MonCat** consisting of monoidal categories, monoidal functors and monoidal transformations. Likewise, there are 2-categories **BrMonCat** and **SymmMonCat**. But I haven't even defined a 2-category yet, so let's stop here!