N-Categories

1/10/02

- want to categorify notion of group (called 2-group)
- A 2-grp will be a category

- 2-groups will be good for parallel transport on 2-manifolds

- more generally than N-categories are

ω-categories

Defn: An ω-category has:

- objects (0-morphisms) $\bullet$
- morphisms (1-morphisms) $sf \bullet \xrightarrow{f} \bullet\, tf$

(here we'd stop if we were just talking about normal categories)

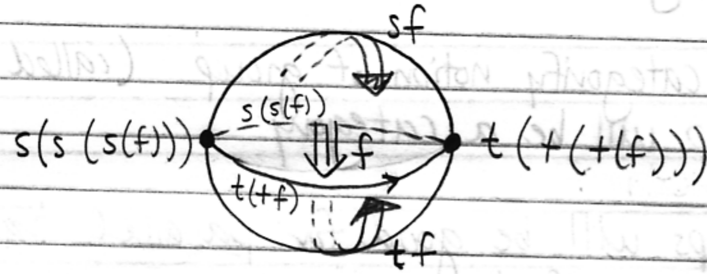
- 2-morphisms

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

A morphism f has a source and target (sf, tf)

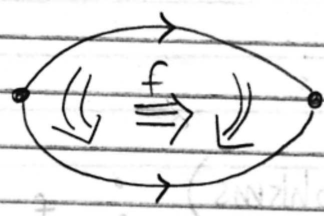
$$\begin{tikzcd} s(s(f)) = s(t(f)) \arrow[r, bend left, "sf"] \arrow[r, bend right, "tf"'] \arrow[r, phantom, "\Downarrow f"] & t(s(f)) = t(t(f)) \end{tikzcd}$$

3-morphisms

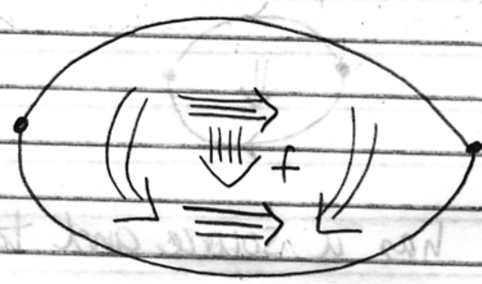


arrows coming around sphere are 2-morphisms

or – we can draw a 3-morphism as



4-morphisms :



Defn: A <u>globular set</u> C is $C_0, C_1, C_2, \ldots$

where $f \in C_n$ is a "n-globe" or "n-cell"

and functions

$$\begin{tikzcd} C_0 & C_1 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] & C_2 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] & C_3 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] & C_4 \arrow[l, bend right, "s"'] \arrow[l, bend left, "t"] \ldots \end{tikzcd}$$

s.t.

$$s(s(f)) = s(t(f))$$

$$t(t(f)) = t(s(f))$$

An <u>ω-category</u> (strict or weak) will be a globular set C with a bunch of "globe-glomming" operations such as:

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[r, bend left, "g"] & z \end{tikzcd} \quad \text{composition giving:} \quad \begin{tikzcd} x \arrow[r, bend left, "fg"] & z \end{tikzcd}$$

"composition"

In a 2-category: (we have many ways of composing)

$$\begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=70, "x", ""{name=X, below}] \arrow[r, "y" description, ""{name=Y}] \arrow[r, bend right=70, "z"', ""{name=Z, above}] & \bullet \arrow[Rightarrow, from=X, to=Y, "f"] \arrow[Rightarrow, from=Y, to=Z, "g"] \end{tikzcd} \quad \text{gives} \quad \begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=50, "x", ""{name=X, below}] \arrow[r, bend right=50, "z"', ""{name=Z, above}] & \bullet \arrow[Rightarrow, from=X, to=Z, "fg"] \end{tikzcd}$$

"<u>vertical</u> <u>composition</u>"

or — "<u>horizontal</u> <u>composition</u>"

$$\begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=50, "a", ""{name=A, below}] \arrow[r, bend right=50, "c"', ""{name=C, above}] & \bullet \arrow[r, bend left=50, "b", ""{name=B, below}] \arrow[r, bend right=50, "d"', ""{name=D, above}] & \bullet \arrow[Rightarrow, from=A, to=C, "f"] \arrow[Rightarrow, from=B, to=D, "g"] \end{tikzcd} \quad \text{giving} \quad \begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=50, "ab", ""{name=X, below}] \arrow[r, bend right=50, "cd"', ""{name=Y, above}] & \bullet \arrow[Rightarrow, from=X, to=Y, "f\cdot g"] \end{tikzcd}$$

Ex)

$$\begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=50, "a", ""{name=A, below}] \arrow[r, bend right=50, "b"', ""{name=B, above}] & \bullet \arrow[r, "c"] & \bullet \arrow[Rightarrow, from=A, to=B, "f"] \end{tikzcd} \quad \text{giving} \quad \begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=50, "ac", ""{name=X, below}] \arrow[r, bend right=50, "bc"', ""{name=Y, above}] & \bullet \arrow[Rightarrow, from=X, to=Y, "f\cdot c"] \end{tikzcd}$$

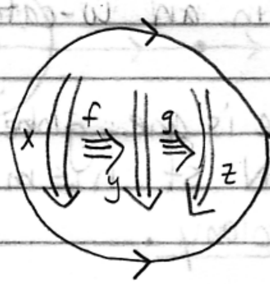
"<u>whiskering</u>"

$$\begin{tikzcd}[column sep=huge] \bullet \arrow[r, "c"] & \bullet \arrow[r, bend left=50, "a", ""{name=A, below}] \arrow[r, bend right=50, "b"', ""{name=B, above}] & \bullet \arrow[Rightarrow, from=A, to=B, "f"] \end{tikzcd} \quad \text{giving} \quad \begin{tikzcd}[column sep=huge] \bullet \arrow[r, bend left=50, "ca", ""{name=X, below}] \arrow[r, bend right=50, "cb"', ""{name=Y, above}] & \bullet \arrow[Rightarrow, from=X, to=Y, "c\cdot f"] \end{tikzcd}$$

"<u>whiskering</u>"

3-cells have 3 Kinds of composition:

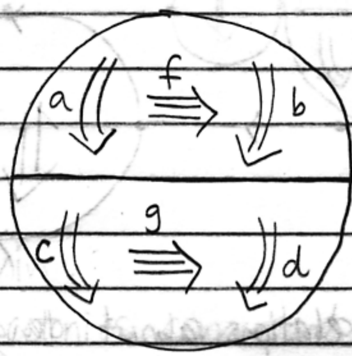
(1)



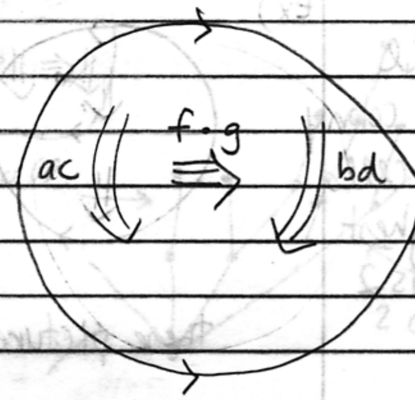
giving

arrows point toward something 1 dimension down

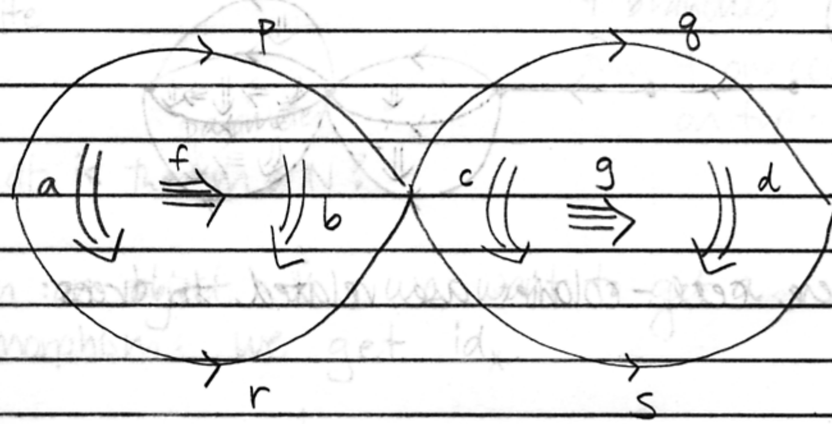
(2)



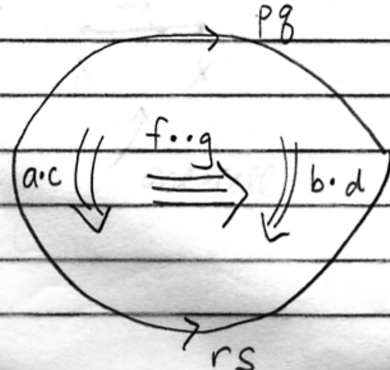
giving



(3)



giving



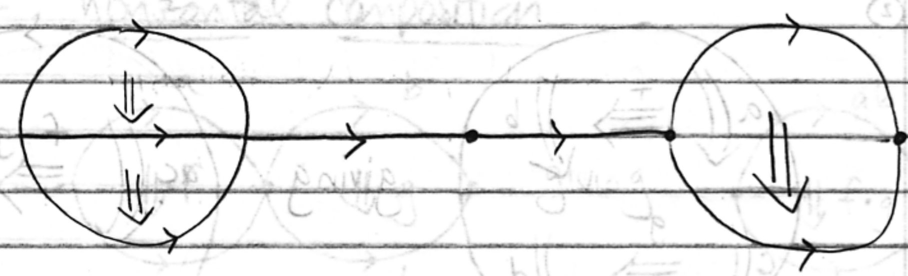
M. Batanin figured out a notation for <u>all</u> globe-glomming operations in an ω-category.

In a strict ω-category there is one globe-glomming operation for each $n, m \in \mathbb{N}$ st $n \leq m$ and each <u>n-dimensional</u> <u>cell</u> <u>colony</u>.

Exs of cell-colonies: (cell colony – these patterns which give relationships of sources/targets)

Ex)

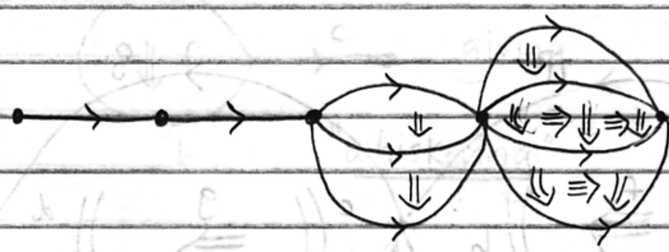
2-dim'l cell colony since cells of highest dim is 2



these pictures tell us relations about targets & sources

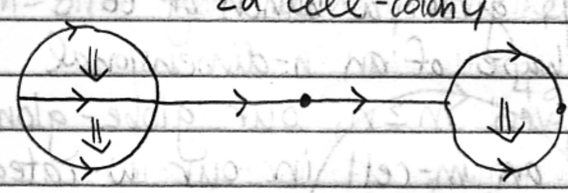
Ex)

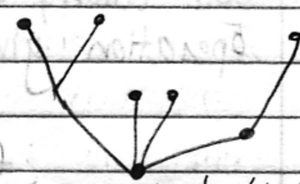
3-d cell colony



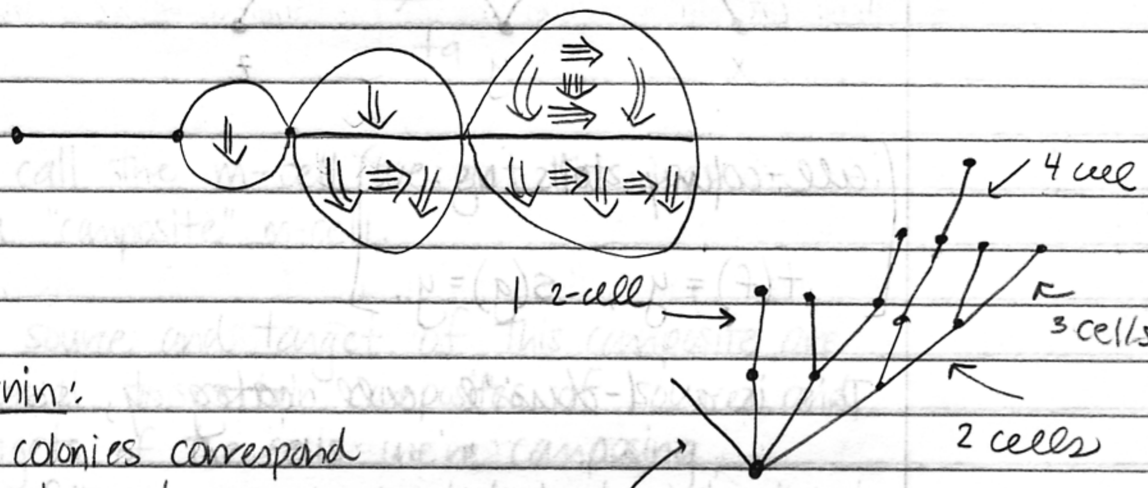
these cell-colonies are related to trees:

2d cell-colony





Ex)



Batanin:
cell colonies correspond to finite planar trees

What is the parameter $m \in \mathbb{N}$?

Given an object x, we want to get a morphism: we get id_x.

$$\dot{x} \qquad x \xrightarrow{1_x} x$$

zero cell

one-cell

In a strict ω-category, given a bunch of cells arranged in the shape of an n-dimensional cell colony and given $m \geq n$, our globe-glomming operation gives us an m-cell in our ω-category.

$n = 1$

$$\begin{tikzcd} x \arrow[r, bend left, "f"] & y \arrow[r, bend left, "g"] & z \end{tikzcd}$$

$x, y, z \in C_0$
$f, g \in C_1$

[cell-colony tells us:

$t(f) = y, \quad s(g) = y.$]

This is a 1-dim'l cell category, so

$m=1$ gives a 1-cell

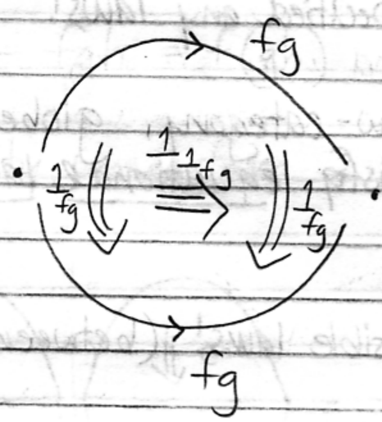
$$\begin{tikzcd} x \arrow[r, bend left, "fg"] & z \end{tikzcd}$$

$m=2$ gives a 2-cell

identity cell

$$\begin{tikzcd} x \arrow[r, bend left=60, "fg"] \arrow[r, bend right=60, "fg"'] \arrow[r, phantom, "\Downarrow 1_{fg}"] & y \end{tikzcd}$$

$m=3$ get identity of the identity of fg

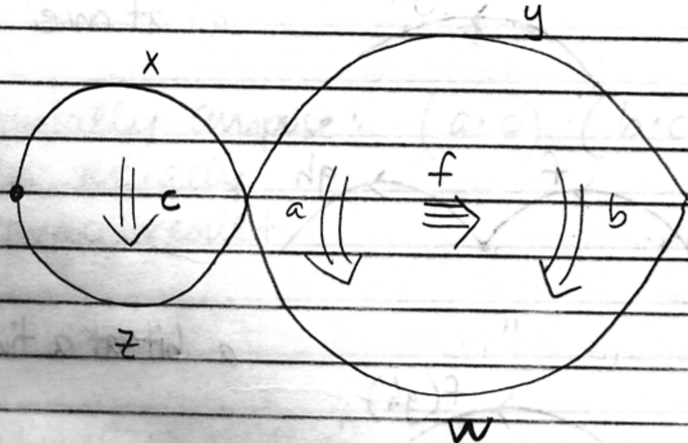


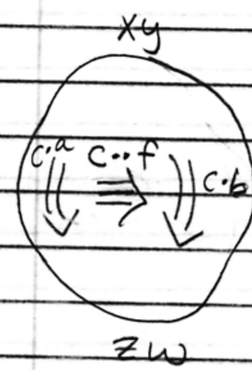
We call the m-cell we get this way a "composite" m-cell.

The source and target of this composite are equal to certain composites of sources and targets of the cells we're composing, following a rule too tedious to describe here

e.g.

$n=3$
$m=3$
(output)





$c \cdot\cdot f : c \cdot a \longrightarrow c \cdot b$

imagine f arrow going into the board

$c \cdot a$: horiz composite

We've basically defined an w-category but we haven't specified any laws!

In a strict w-category, globe-glomming operations satisfy <u>equational laws</u> (like assoc, etc) (this is evil)

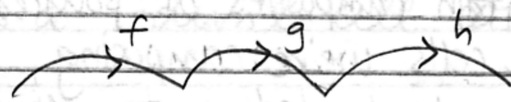
What are they?

* They are all possible laws! (between globe glomming operations)

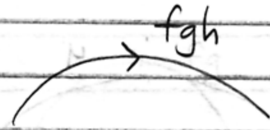
We can glom globes <u>all at once</u> or a <u>bit at a time</u> and we always get the same result.

ie – have more than one way of doing something, do any way and get same thing.

Ex)

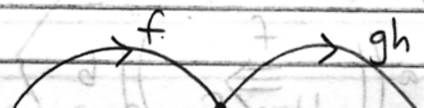


gives



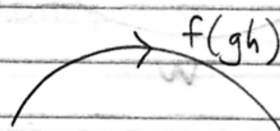
all at once

or



=

a bit at a time.

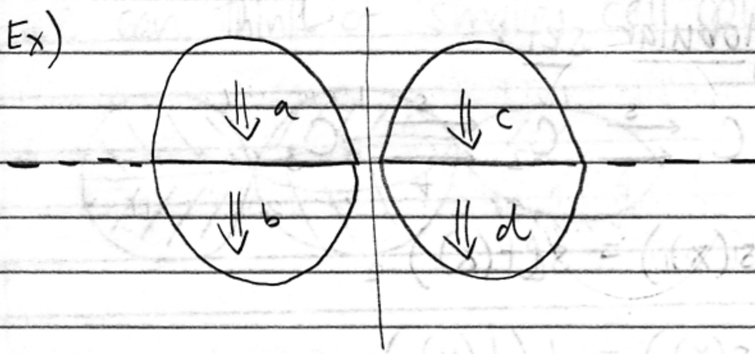


ie) get half assoc. law

$$(fg)h = fgh = f(gh)$$

so - associativity comes from 2 "half-associativity" laws.

Ex)



| vertically compose: $(ab)\cdot(cd)$

$= $ all at once

$=$

--- horizontally compose: $(a\cdot c)\ (b\cdot c)$
then vertically
compose result