1/14/02

Note — a group is a category w/ one object and all morphisms invertible.

<u>Summary</u>: A strict ω-category C is a <u>globular</u> <u>set</u>:

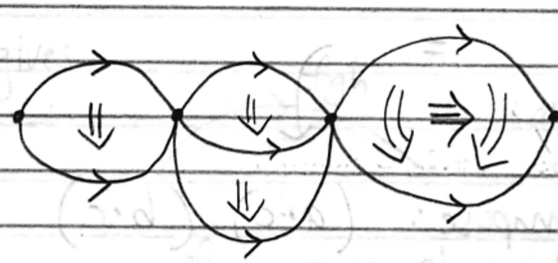
$$C_0 \underset{t}{\overset{s}{\leftleftarrows}} C_1 \underset{t}{\overset{s}{\leftleftarrows}} C_2 \underset{t}{\overset{s}{\leftleftarrows}} C_3 \ldots$$

st $\quad s(s(x)) = s(t(x))$

$$t(s(x)) = t(t(x))$$

equipped w/ operations: one for each n-dim'l cell colony:

$n = 3$

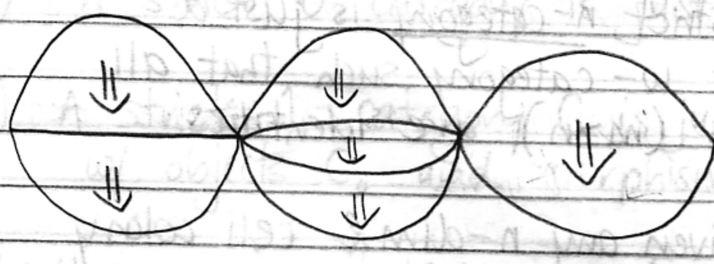


and each $m \geq n$.

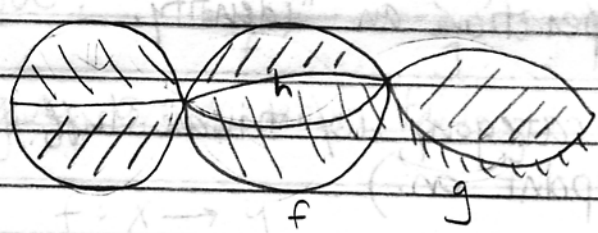
For this, we have operations for $m = 3, 4,$ etc.

These operations satisfy all possible equational laws:

this is a 3d cell colony

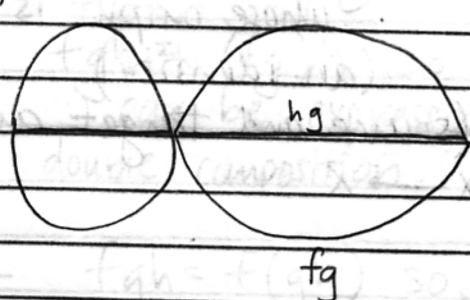


we can think of smaller cell colonies living in here.



we could glom them all together, or in groups.

In groups:



<u>Thm</u>: And the results are the same (grouping all together or in groups)!

ie) Given a cell colony built in a legal fashion from smaller cell colonies, the operation corresponding to big colony gives same output as result of first composing each of the little colonies and then composing the result.

Defn: A strict n-category is just a strict ω-category such that all m-cells ($m>n$) are identities.

Recall – given any n-dim'l cell colony and any $m \geq n$ we get an operation; for $m \geq n$ we call the result of doing this operation an "identity."

(it's an ω-category w/ trivial stuff from some point on.)

Ex) $n=0$ $\underset{x}{\bullet}$ $m=0$ gives an operation whose output is x

we call these identities

$n=0$ $\underset{x}{\bullet}$ $m=1$ gives an operation whose output is 1_x (an identity) (source and target are $\underset{x}{\bullet}$)

$$1_x : x \longrightarrow x$$

$n=0$ $\underset{x}{\bullet}$ $m=2$ gives a 2-cell gives an operation w/ output

$$1_{1_x} : 1_x \Longrightarrow 1_x$$

ex) A strict 0-category C is just a set. (C_0)

ex) A strict 1-category C is just a category w/ objects C_0 and morphisms C_1.

where assoc & unit laws came from:

<u>assoc</u>: $\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet$ all colony

gives an operation taking

$$\begin{array}{l} f: x \to y \\ g: y \to z \\ h: z \to w \end{array} \quad \text{to} \quad fgh: x \to w$$

doing 3 individually is same as glomming all together.

ie) $fgh = (fg)h$ so this "triple composition" can be expressed in terms of double composition.

Also — $fgh = f(gh)$ so, together we get assoc. law.

Ex) Every strict n-category gives a strict ω-category.

Ex) Chain complexes: law $\partial\partial = 0$.
define $\partial = t - s$
Then we get $\partial\partial = 0$.

so every chain complex gives a strict ω-category (generalizes homology theory)

Batanin's Work

Weak ω-categories

Defn: A *weak ω-category* C is a globular set, equipped w/ operations.
For starters (we have same operations as before).
That is –
there is an operation for each n-dim'l cell colony and each $m \geq n$.
(now, we won't have equality when doing things in a different order, we have isomorphisms).

say we end up w/ 2 m-cells, we get an m+1 cell which is an operation taking us from one m-cell to other.

But now – we keep recursively throwing in new operations: whenever we have a big cell colony built from little ones in a legal fashion and we can use our existing operations to get a new m-cell from this data all at once (using 1 operation) or in stages (using lots of operations), we get 2 NEW operations producing an $(m+1)$ cell from the same data, going from the result of "all-at-once" to "a bit at a time," and vice-versa.

Defn: A *weak n-category* is a weak ω-category with only identity m-cells ($m > n$).
(equal)
*so this is strict on top.

Ex) Strict vs weak 2-categories.

<u>Strict 2 categories:</u>

objects C_0 (0-cells)
morphisms C_1 (1-cells)
2-morphisms C_2

$\underline{C_1}$

If
$s(f) = x$
$t(f) = y$ we write

$f : x \longrightarrow y$

In C_2 $\alpha : f \Rightarrow g$

<u>Cell colonies</u>

$n = 0$ $\bullet$

$m = 0$ given $x \in C_0$ this gives x

$m = 1$ given $x \in C_0$ this gives $1_x : x \longrightarrow x$

$n = 1$ $\bullet \longrightarrow \bullet$

$m = 1$ * given $f : x \longrightarrow y$ this gives f.

$m = 2$ ** given $f : x \to y$ this gives $1_f : f \Rightarrow f$ 2 cell

$m = 3$ gives id of id

* is identity
** : takes f and makes an identity

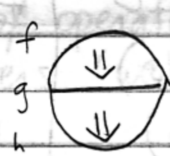
<u>Note</u>: 1_f <u>not</u> a morphism in category

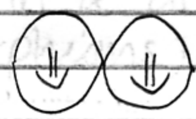
None of these operations are morphisms in the category.

$n=1$ $\bullet \longrightarrow \bullet \longrightarrow \bullet$ $m=1$ given $f: x \to y$, $g: y \to z$
gives $fg: x \to z$

cell colony

$n=2$  $m=2$ given $\alpha: f \Rightarrow g$
this gives α.

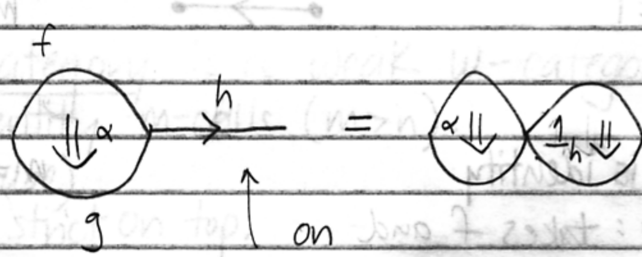
 $m=2$ given $\alpha: f \Rightarrow g$
$\beta: g \Rightarrow h$
this yields
$\alpha\beta: f \Rightarrow h$
(vertical comp)

 $m=2$ given $\alpha: f \Rightarrow g$
$\beta: f' \Rightarrow g'$

$f, g: x \longrightarrow y$
$f', g': y \longrightarrow z$
this yields
$\alpha \cdot \beta: ff' \Rightarrow gg'$
(horizontal comp)

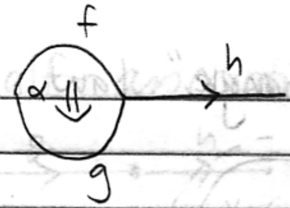
Other operations can all be expressed in terms of these, e.g.:

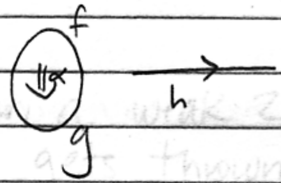
(3-comp expressed as 2-comp)

whiskering 

want to express whiskering in terms of horizontal comp (which we already have)

on this 1-cell we use the operation which produces the 2-cell (1_h)

We break up  as

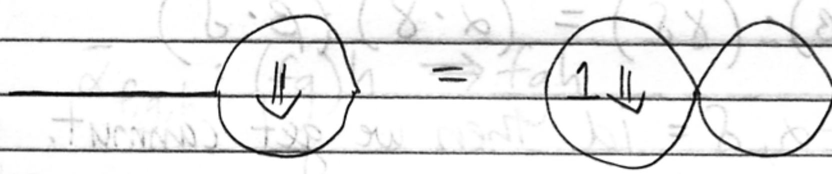


use operation that is identity

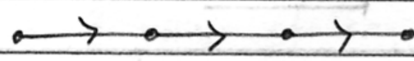
use operation that produces the 2-cell 1_h

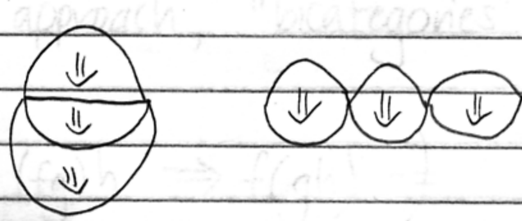
then use horizontal composition.

Similarly —

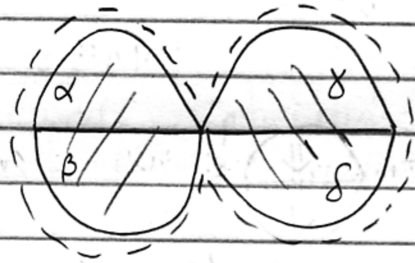


Interesting laws:

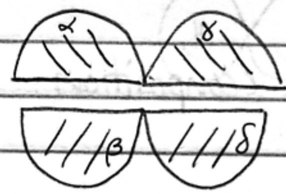
- left/right unit laws for morphisms, 2-morphisms (vert & horiz comp)
- assoc. 



• "middle 4 interchange" law



can be chopped up as vertically or



horizontally.

$$(\alpha\beta)\cdot(\gamma\delta) = (\alpha\cdot\gamma)(\beta\cdot\delta)$$

if $\alpha, \delta = id$ then we get commut.

end strict 2-categories

Weak 2-categories

• all these laws are eliminated and replaced by new operations.

Ex) We have half-associativity laws

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h} \bullet \qquad fgh = (fg)h$$

So, in a weak 2-category, the half-associativity law gets thrown out; instead we have a new <u>operation</u> which takes 3 1-morphisms:

$$\xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h}$$

and gives a 2-morphism

$$\alpha_{f,g,h} : fgh \Rightarrow (fg)h$$

the "half-associator" and also

$$\bar{\alpha}_{f,g,h} : (fg)h \Rightarrow fgh$$

Also –

$$\beta_{f,g,h} : fgh \Rightarrow f(gh)$$

f, g, h are composable 1-morphisms

$$\bar{\beta}_{f,g,h} : f(gh) \Rightarrow fgh$$

{In an older approach, "bicategories" we have associator:

$$A_{f,g,h} : (fg)h \Rightarrow f(gh) \}$$

↑ this is $\beta \circ \bar{\alpha}$ in our approach.

Note: In a weak 2-category we don't have any 3-morphisms (they're all trivial).

In a weak ω-category we have operation

(this is an equivalence)

$$X_{f,g,h} : \alpha_{f,g,h} \bar{\alpha}_{f,g,h} \Rrightarrow 1_{f,g,h}$$

but in a weak 2-category, mercifully, this is an identity 3-morphism, so in our weak 2-category we get $\bar{\alpha} = \alpha^{-1}$.

so α, $\bar{\alpha}$ wouldn't be strict inverses but there is a process (operation) taking $\alpha\bar{\alpha}$ to identity.