1/16/02

We've seen that in a weak $\omega$-category, associativity doesn't hold "on the nose." (in fact nothing holds "on the nose")

We've seen that given

$$\bullet \xrightarrow{a} \bullet \xrightarrow{b} \bullet \xrightarrow{c} \bullet \qquad a, b, c \in C$$

we get three kinds of "composite"

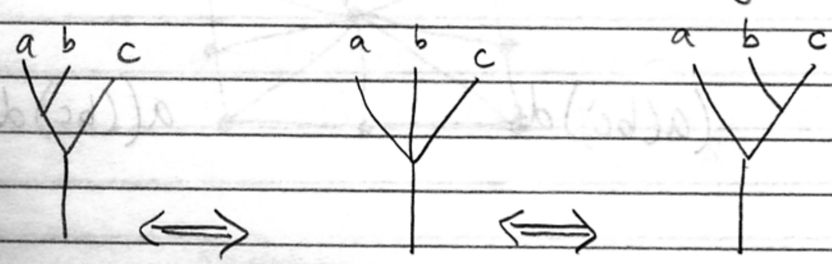
$$(ab)c \Longleftrightarrow abc \Longleftrightarrow a(bc)$$

These aren't equal, but we have a 2-morphism going between them.

they are related by 2-morphisms, "$\Longleftrightarrow$" the "half associators" and their "weak inverses".

$\Rightarrow \circ \Leftarrow$ isn't the identity (there is a 3-morphism
(composition) from the composite to the identity)

Pictorially: these 3 ways of composing are:



<u>Note</u>: In weak 1-category (a real category) all the 2-morphisms on prev pg are identities (equals).

Recall – a weak n-category is a weak n-category where all morphisms above n are trivial.

(In a weak 1-category the 2-morphisms here are <u>identities</u> so $(ab)c = abc = a(bc)$).

Next consider:

$$\xrightarrow{a}\bullet\xrightarrow{b}\bullet\xrightarrow{c}\bullet\xrightarrow{d}\bullet \qquad a,b,c,d \in C_1$$

There are many ways to chop this up and group together.

Corresponding to the decompositions of this into smaller cell colonies we get various ways of composing $a,b,c,d$:

$(ab)(cd)$

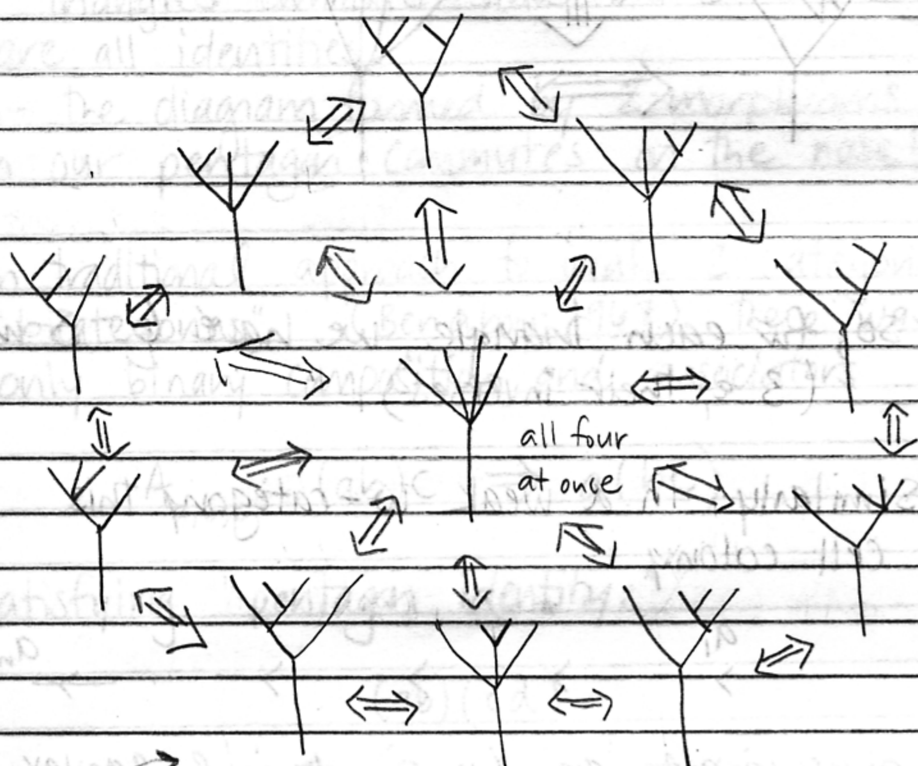
$((ab)c)d$ $\qquad\qquad$ $a(b(cd))$

$(a(bc))d$ $\qquad\qquad$ $a((bc)d)$

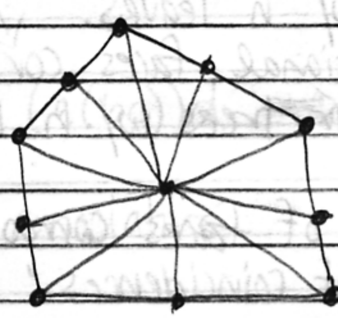
These five came from repeated binary composition. There are more:

In tree notation:

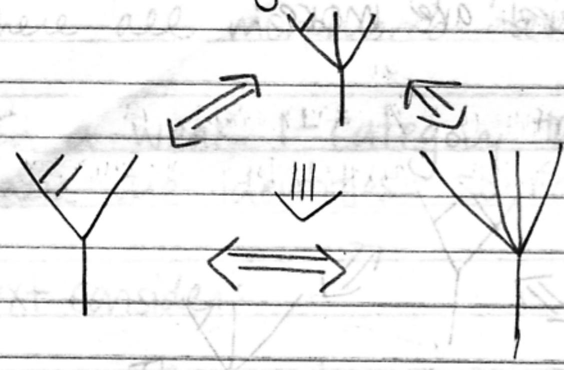
ternary trees between 2 binary trees




This is <u>barycentric</u> <u>subdivision</u> of a pentagon.



we have triangles



so, for each triangle, we have 6 3-morphisms (3 & their inverses)

Similarly, in a weak $\omega$-category this cell colony

$$\xrightarrow{a_1} \to \to \to \cdots \xrightarrow{a_n}$$

(prev. pg) We had a,b,c,d and ended up w/ a 2-dim'l shape.

gives rise to an $(n-2)$ dim'l convex polytope, the "associahedron",

whose vertices correspond to binary planar trees w/ $n$ leaves.
Higher-dimensional faces correspond to all other planar trees w/ $n$ leaves,
with dimensions of faces corresponding to "number of coincidences."

James Stasheff discovered these (~1968) while working on topology.

Note: If we're in a weak 2-category, all those triangles commute since the 3-morphisms are all identities.

So- the diagram formed by 2-morphisms in our pentagon commutes on the nose!

In traditional approach to weak 2-categories "bicategories," (Benabou 1967) there was only binary composition and associators

$$A_{a,b,c} : (ab)c \Rightarrow a(bc)$$

satisfying "pentagon identity."

$$\begin{tikzcd}[column sep=small]
 & (ab)(cd) \arrow[dr, Rightarrow] & \\
((ab)c)d \arrow[ur, Rightarrow] \arrow[d, Rightarrow] & & a(b(cd)) \\
(a(bc))d \arrow[r, Rightarrow] & a((bc)d) \arrow[ur, Rightarrow] &
\end{tikzcd}$$

built up w/ associators

Examples:

① There's a weak $\omega$-cat called Top where objects are topological spaces, 1-morphisms are continuous functions between them

$$\begin{tikzcd} X \arrow[r, bend left=60, "f"] \arrow[r, bend right=60, "g"'] \arrow[r, phantom, "\alpha \overset{T}{\Rrightarrow} \beta"] & Y \end{tikzcd}$$

$X, Y$ top spaces
$f, g$ cont functs

continuously deform $f$ to $g$

2-morphisms $\alpha : f \Rightarrow g$ are homotopies

$$\alpha : X \times [0,1] \longrightarrow Y$$

s.t.

$$\alpha\big|_{X \times \{0\}} = f \qquad \alpha\big|_{X \times \{1\}} = g$$

3-morphisms $\quad T : \alpha \Rrightarrow \beta$

are homotopies between homotopies:

$$T : X \times [0,1]^2 \longrightarrow Y$$

That is, at each step we have a homotopy

st $\forall\ t \in [0,1]$

$T(\underset{X}{-}, \underset{[0,1]}{-}, t)$ is a homotopy from $f$ to $g$, and equals $\alpha$ when $t=0$
$\beta$ when $t=1$

; etc.

Given a weak $\omega$-category $C$ and objects $x, y \in C_0$ we can define a weak $\omega$-category "$\underline{\hom(x,y)}$".

(If $C$ is an $n$-category, $\hom(x,y)$ is an $(n-1)$ category.)

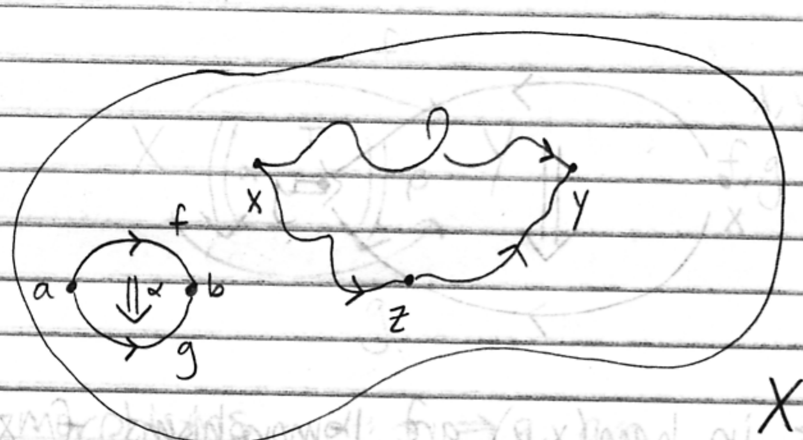
$$\begin{tikzcd} x \arrow[r, bend left=50] \arrow[r, bend right=50] \arrow[r, phantom, "\Downarrow"] & y \end{tikzcd}$$

- Objects in $\hom(x,y)$ are 1-morphisms $f : x \to y$
- morphisms $\alpha : f \to g$ in $\hom(x,y)$ are 2-morphisms $\alpha : f \Rightarrow g$ in $C$, $f, g : x \to y$

(Similarly, $n$-morphisms in $\hom(x,y)$ are $(n+1)$-morphisms in $C$).

Repeatedly applying this, given $k$-morphisms $f, g : x \xRightarrow{k \text{ lines}} y$ in $C$ we get $\hom(f,g)$, which is an $\omega$-cat whose $n$-morphisms will be $(n+k+1)$-morphisms in $C$.
These are "microcosms."

Example: Given a topological space $X$ and let $*$ be the one-point space; we get a microcosm $\hom(*, X)$ which is a weak $\omega$-category.




objects in $\hom(*, X)$ are points.

morphisms are paths, 2-morphisms are homotopies

$\hom(*, X)$ is called (by Grothendieck in his 600 pg letter to Quillen).
the <u>fundamental</u> <u>$\omega$-groupoid</u> of $X$, $\Pi(X)$.

Roughly, a weak $\omega$-groupoid is a weak $\omega$-category st every $j$-morphism ($j > 0$) is an equivalence.

Recall: We have a homotopy between the id.  and this path

An equivalence is a j-morphism $f: x \longrightarrow y$ w/ a j-morphism $\bar{f}: y \rightarrow x$ which is an inverse up to equivalence ie) there are equivalences

$$\alpha : f\bar{f} \Rightarrow 1_x$$

$$\beta : \bar{f}f \Rightarrow 1_y$$

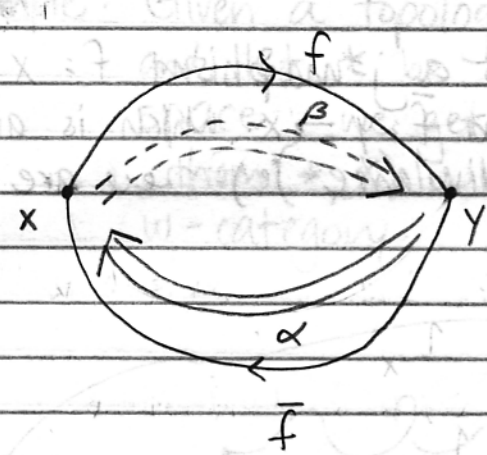
(for n-categories, after n they're all identities.)

This is a well-formed defn. for a weak n-category if we declare identities are equivalences.

In general we demand that f has a "weak inverse" $\bar{f}$ ie) $\exists$ $\alpha : f\bar{f} \Rightarrow 1$, $\beta : \bar{f}f \Rightarrow 1$ st $\alpha, \beta$ have $\gamma, \delta, \varepsilon, \eta$

$$\gamma : \alpha\bar{\alpha} \Rightarrow 1 \qquad \delta : \bar{\alpha}\alpha \Rightarrow 1$$
$$\varepsilon : \beta\bar{\beta} \Rightarrow 1 \qquad \eta : \bar{\beta}\beta \Rightarrow 1$$

do this forever (the work doubles at each stage)
(next step — $\gamma, \delta, \varepsilon, \eta$ each have equivalences between them, and so on)




w/ $\bar{\alpha}, \bar{\beta}$ going opposite ways

$S^\infty$ — contractible