Structure Theorems 1/24/02

- A groupoid is a category with all morphisms invertible. (Here, strict = weak since we have no 2 morphisms.)

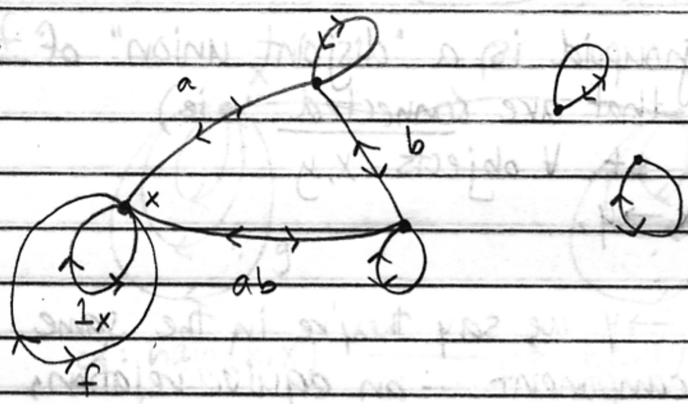
Note: morphisms must satisfy a bunch of laws composition must be assoc., left/right unit laws.

* Everything holds on the nose.

- every morphism has an inverse.

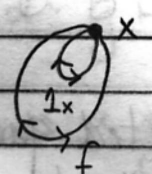
A groupoid:

consists of pieces



$f^2 = 1_x \Rightarrow f = f^{-1}$

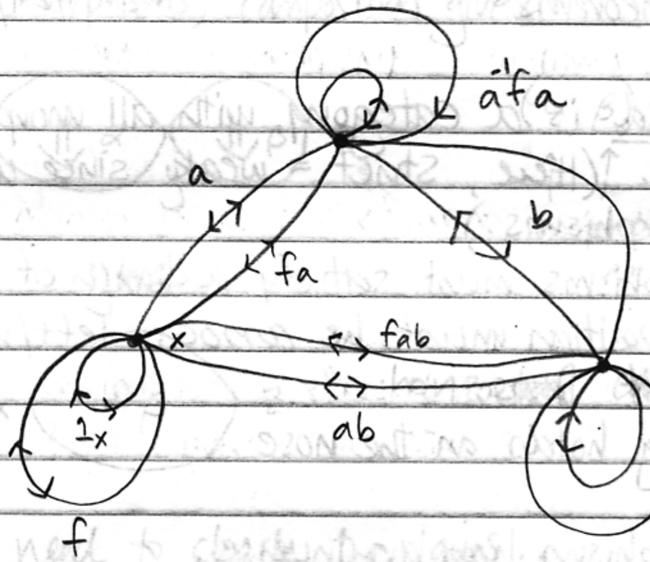
A group is a groupoid w/ one element.



this is the group $\mathbb{Z}_2$ w/ 2 elts: 1_x, f

We need to compose f w/ a and ab.

$f = f^{-1}$



$$a^{-1}fab = (fa)^{-1}ab$$
$$= a^{-1}f^{-1}ab$$
$$= a^{-1}fab$$
since $f^{-1} = f$

So – every groupoid is a "disjoint union" of groupoids that are <u>connected</u>; ie) groupoids st $\forall$ objects x, y $\exists\ f : x \to y$.

If $\exists\ f : x \to y$ we say they're in the same connected component – an equiv. relation, since (among other things) if $f : x \to y$ then $f^{-1} : y \to x$.

Note – we use notion of groupoid for symmetric property

"disjoint union" – no morphism going from one groupoid to other.

So – it suffices to study connected groupoids.

<u>Note</u> – A connected <u>component</u> of x in a groupoid is a groupoid consisting of all objects y st $\exists\ f : x \to y$ (fixing x) <u>and</u> all morphisms between them.

Let C be a connected groupoid w/ nonempty set S of objects.

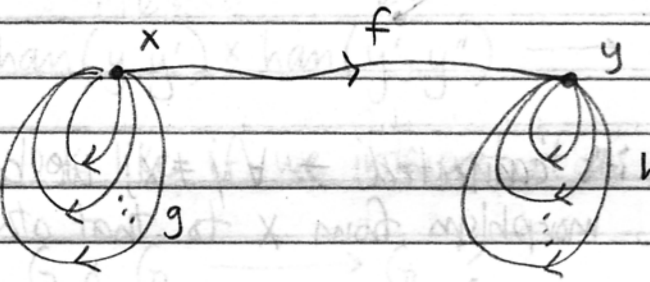
Pick $x \in S$. Then $\mathrm{hom}(x,x)$ is a group G.

But — what if we chose some other point y. Then $\mathrm{hom}(y,y) \cong \mathrm{hom}(x,x)$ (look at picture on last pg — all pts have group $\mathbb{Z}_2$).

$$\mathrm{hom}(x,x) \cong \mathrm{hom}(y,y)$$

not isomorphic in a unique way

pf:



$$\alpha : \mathrm{hom}(x,x) \longrightarrow \mathrm{hom}(y,y)$$

Choose $f : x \to y$.

Then for $g \in \mathrm{hom}(x,x)$

$$\alpha : g \longmapsto f^{-1} g f$$

Then α is a group homo

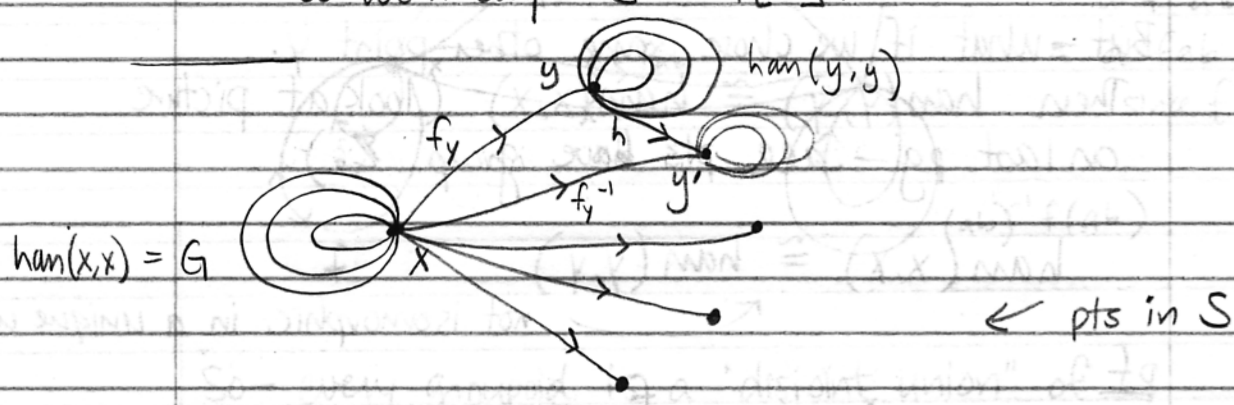
$$f^{-1}(gg')f = (f^{-1}gf)(f^{-1}g'f)$$

α is 1-1 and onto since

$$\alpha^{-1} : h \longmapsto f h f^{-1}$$

So— groupoid has groups in it:
all $hom(x,x)$ endomorphisms
(morphisms from x to itself)

<u>Claim</u>: G and S determine C up to isomorphism,
so we'll say $C \cong G[S]$.



Assuming it's connected — $\forall y \neq x$, we can find a morphism from x to that other pt.

$\forall\ y \neq x$ in S, choose $f_y : x \to y$.

These f_y's give isomorphisms

$$\alpha_y : hom(x,x) \xrightarrow{\sim} hom(y,y)$$

as we did before. We get maps from y to itself by doing f_y^{-1}(something in $hom(x,x)$) f_y.

If $y = x$, choose $\alpha_y = 1_x$ (identity)

Now note $\forall\, y, y' \in C$

$$\alpha_{yy'} : \mathrm{hom}(y,y') \xrightarrow{\sim} \mathrm{hom}(x,x) = G$$

$$h \longmapsto f_y h f_{y'}^{-1}$$

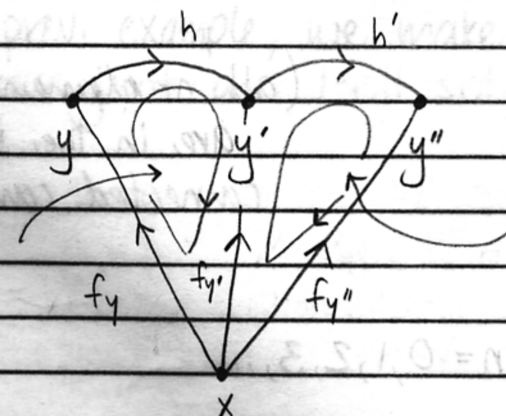
so now we have an iso bet. any hom set $\mathrm{hom}(x,y)$ and $\mathrm{hom}(x,x) = G$.

What does

$$\circ : \underset{h}{\mathrm{hom}(y,y')} \times \underset{h'}{\mathrm{hom}(y',y'')} \longrightarrow \underset{hh'}{\mathrm{hom}(y,y'')}$$

look like if we interpret it as a map

Quest: $G \times G \longrightarrow G$?



composition in here is $\alpha_{yy'} h$

composition in here is $\alpha_{yy'} h'$

~~*check this~~

Answer: $(\alpha_{yy'} h, \alpha_{y'y''} h') \longmapsto (\alpha_{yy'} h)(\alpha_{y'y''} h')$

$G \times G \longrightarrow G$ multiplication in G!

<u>Structure</u> <u>Theorem</u>: A group G and a set S determine a connected groupoid $G[S]$ w/ S as objects and $\hom(x,x) \cong G \quad \forall x \in G$ up to <u>isomorphism</u>.

canonical if we pick $x \in S$.

Any groupoid is isomorphic to a disjoint union of groupoids of this form:

disjt union

$$\coprod_{\alpha} G_{\alpha}[S_{\alpha}].$$

these are the connected components

<u>Example</u>: let C be the groupoid whose objects are finite sets and morphisms are all 1-1 and onto functions (we want them to be invertible).

The components of C are:

- 1-elt sets
- 2-elt sets
- 3-elt sets
- ⋮ etc

(all n-element sets are in the same connected component)

correspond to $n = 0, 1, 2, 3, \ldots$

so S_n = the set of all n-element sets.

hom(x,x) where $x \in S_n$. So x is a set w/ n elts.

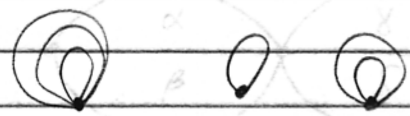
And G_n = the group of permutations of an n-element set "n!" (S_n)

So:

$$C = \coprod_{n=0}^{\infty} n!\,[S_n]$$

our name for the perm. group S_n.

Defn: We call a groupoid skeletal if all components have only one object. (just a bunch of groups).



If C is skeletal, $C = \coprod_{\alpha} G_\alpha$ groups

If in prev. example, we make it skeletal, we have just one 1-elt set, one 2-elt set, etc...

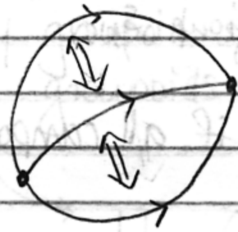
2-Groupoids

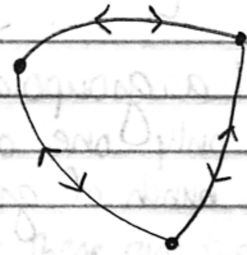
Now we'll just study strict 2-groupoids, i.e. strict 2-categories w/ all morphisms and invertible "on the nose". (Later we'll do weak ones.)

this 2-morphism isn't going from inverse to inverse.

ie) $\alpha : f \Rightarrow g$

but $\alpha : f^{-1} \not\Rightarrow g^{-1}$





If we forget about 2-morphisms, we have a groupoid.

Given a 2-groupoid C, we could ignore the 2-morphisms and get a groupoid $\tilde{C}$.

(We could also decategorify and get a groupoid $C[1]$.)

hom(x,x) has morphisms & 2-morph.

Given $x \in C$, the set of morphisms $f : x \to x$ is a group.

Also, the set of 2-morphisms $\alpha : 1_x \longrightarrow 1_x$ forms a group, under vertical composition.

how does this differ from hom(x,x)? *

$$\begin{tikzcd} x \arrow[r, bend left=60, "1_x"] \arrow[r, "1_x" description] \arrow[r, bend right=60, "1_x"'] & x \end{tikzcd}$$

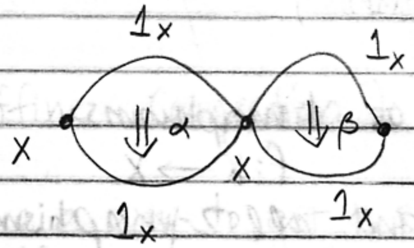
$\Downarrow\alpha$ $\Downarrow\beta$

$\alpha\beta : 1_x \Rightarrow 1_x$

$\overset{\text{2 morphisms from } 1_X \text{ to } 1_X.}{\shortparallel}$

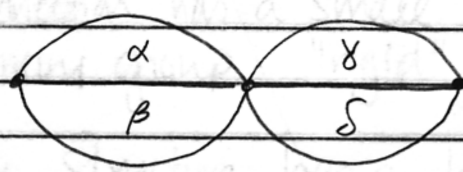
This 2^{nd} group, $\hom(1_X, 1_X)$ is interesting because it is abelian.

Notice– This group also has another product given by horizontal composition:



$\alpha \cdot \beta : 1_X \Rightarrow 1_X$

We also have:

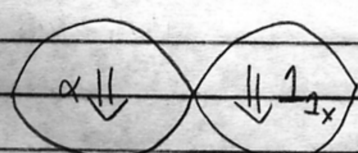


$$(\alpha \cdot \gamma)(\beta \cdot \delta) = (\alpha\beta) \cdot (\gamma\delta)$$

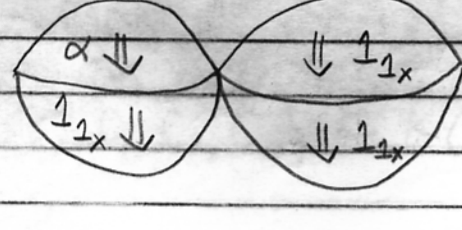
- vert. composition makes this into a group but we don't know that horizontal comp. does. (in fact it does)

* Claim: 1_{1_X} is also identity for horiz. comp.

$\alpha \cdot 1_{1_X}$



$=$



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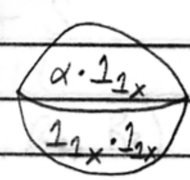

$= (\alpha \cdot 1_{1_X})(1_{1_X} \cdot 1_{1_X})$

So –

$$(\alpha \cdot 1_{1_X}) = (\alpha \cdot 1_{1_X})(1_{1_X} \cdot 1_{1_X})$$

$$\Rightarrow 1_{1_X} \cdot 1_{1_X} = 1_{1_X} \text{ (identity)}$$

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hom(X,X) consists of 1-morphisms $f: X \to X$ and all 2-morphisms between them.