<u>Goal</u>: Classify 2-groups 1/29/02

<u>Thm</u>: Every groupoid is isomorphic to

$$\coprod_\alpha G_\alpha[S_\alpha] \quad \text{for groups } G_\alpha \text{ and sets } S_\alpha.$$

Now – for strict 2-groupoids (2-cat where all morph. & 2-morph are strict on the nose)

- we'll need simplifying assumptions

Let C be a strict 2-groupoid. Let $x \in C_0$ be an object. The set of 1-morphisms $f: x \to x$ form a group, G.

Let A be the set of 2-morphisms
$\alpha: 1_x \Rightarrow 1_x$

- These form a group also!

$$\begin{tikzcd} x \arrow[r, bend left, "1_x"] \arrow[r, bend right, "1_x"'] \arrow[r, phantom, "\Downarrow\alpha"] & x \end{tikzcd}$$

- A becomes a group under vertical composition, w/ identity $1_v = 1_{1_x}$.

- We can also compose α's horizontally.
- A also gets an associative product from horizontal composition

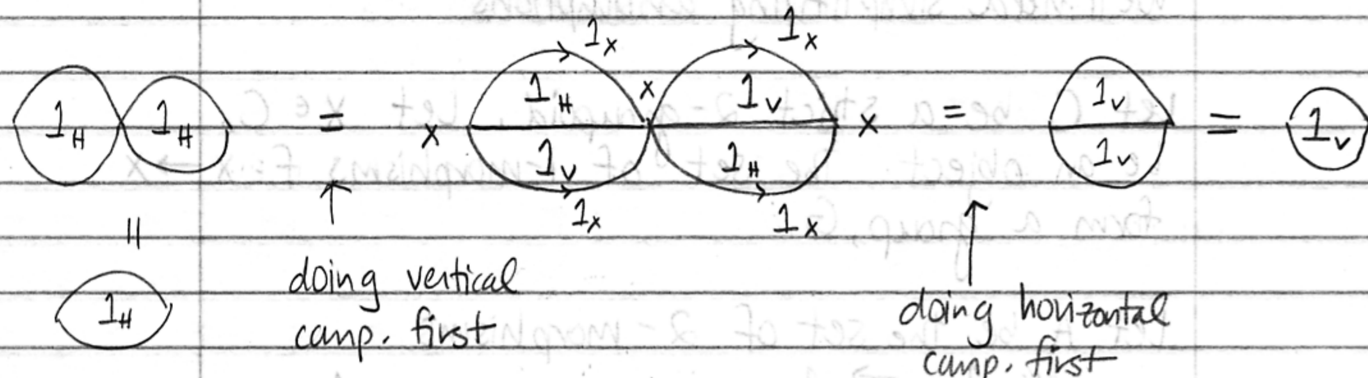
$$\alpha\cdot\beta = \begin{tikzcd} x \arrow[r, bend left, "1_x"] \arrow[r, bend right, "1_x"'] \arrow[r, phantom, "\alpha\Downarrow"] & x \arrow[r, bend left, "1_x"] \arrow[r, bend right, "1_x"'] \arrow[r, phantom, "\beta\Downarrow"] & x \end{tikzcd}$$

* the identity of horiz. comp = identity of vert. comp
$= 1_v$.

* In fact, 1_v is id. for horizontal composition.
But - suppose we only knew there exists an id. for horizontal comp., $1_H : 1_x \Rightarrow 1_x$
Show

$$1_v = 1_H$$

<u>Pf</u>: Uses interchange law:



In fact - vertical & horizontal comp. are the same!

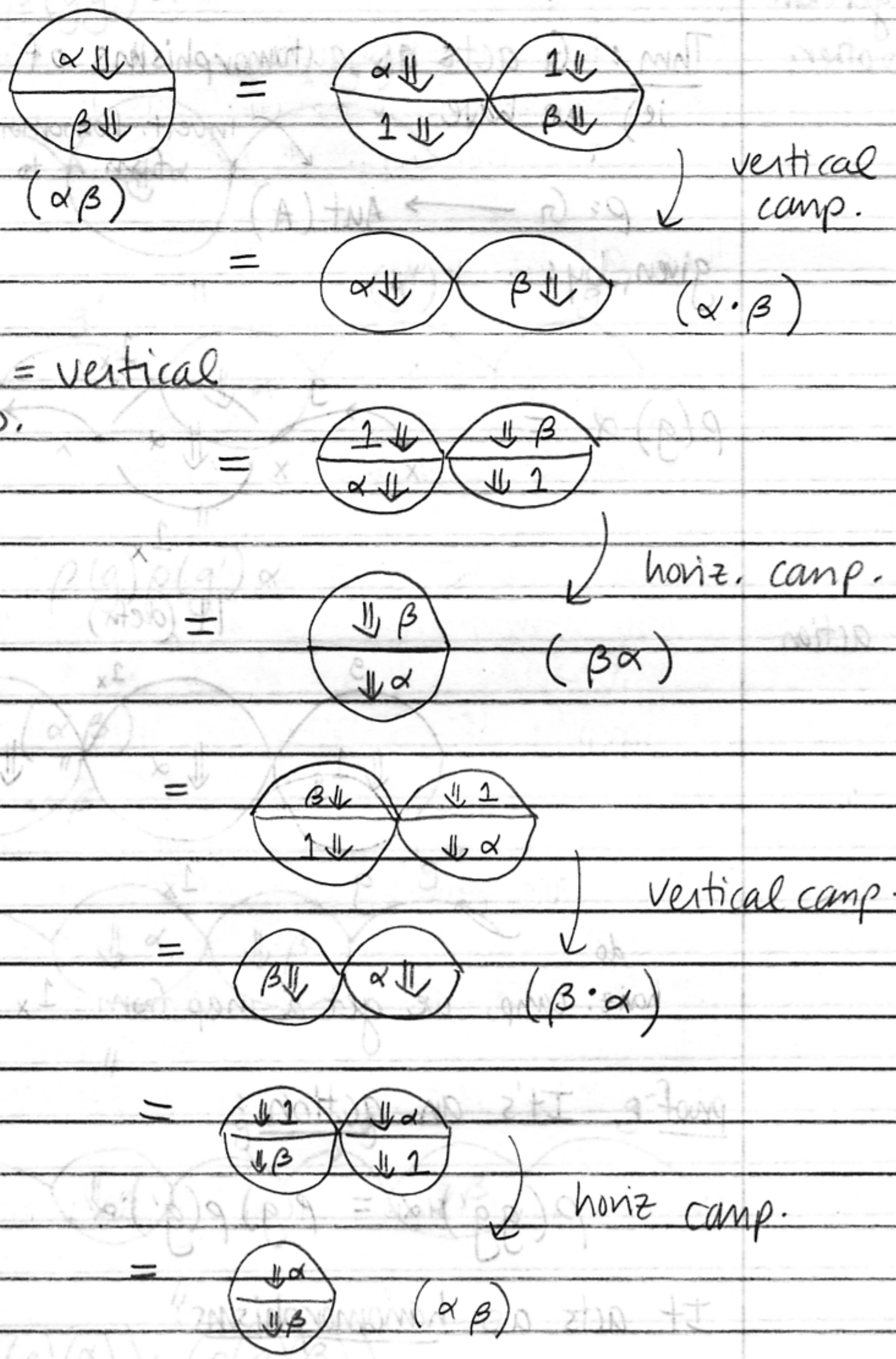
<u>Lemma</u>: Given $\alpha, \beta \in A$

$$\alpha\beta = \alpha \cdot \beta = \beta\alpha = \beta \cdot \alpha$$

<u>Note</u>: This implies A has only one group structure - either horiz. or vert. composition, and this group is abelian.

So — A is an abelian group under horiz = vertical composition!

pf :

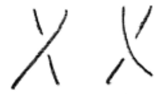


• uses interchange law

so horiz = vertical comp.

* this is called Eckmann – Hilton argument

Note:



are not equivalent. can't get from one to other.

(3-categories - includes mathematics of knot theory)

Thm: G acts as automorphisms of A.
ie) we have

invert. homomorphism from A to itself

$$\rho : G \longrightarrow \mathrm{Aut}(A)$$

given by:

$$\rho(g)\alpha = \begin{tikzcd} x \arrow[r, bend left, "g"] & x \arrow[r, bend left, "1_x"] \arrow[r, bend right, "1_x"'] \arrow[r, phantom, "\Downarrow\alpha"] & x \arrow[r, bend left, "g^{-1}"] & x \end{tikzcd}$$

left action

$$\| \text{ (defn)}$$

$$\begin{tikzcd} x \arrow[r, bend left, "g"] \arrow[r, bend right, "g"'] \arrow[r, phantom, "\Downarrow 1_g"] & x \arrow[r, bend left, "1_x"] \arrow[r, bend right, "1_x"'] \arrow[r, phantom, "\Downarrow\alpha"] & x \arrow[r, bend left, "g^{-1}"] \arrow[r, bend right, "g^{-1}"'] \arrow[r, phantom, "\Downarrow 1_{g^{-1}}"] & x \end{tikzcd} \in A$$

do horiz. comp, we get a map from 1_x to 1_x

proof: It's an action:

$$\rho(gg')\alpha = \rho(g)\rho(g')\alpha$$

It acts as homomorphisms:

$$\rho(g)(\alpha\beta) = \big(\rho(g)(\alpha)\big)\big(\rho(g)(\beta)\big)$$

(automorphisms: $\rho(g)^{-1} = \rho(g^{-1})$)

<u>Check</u>: ① $\rho(gg')\alpha$

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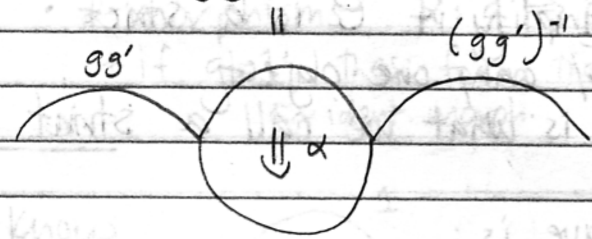

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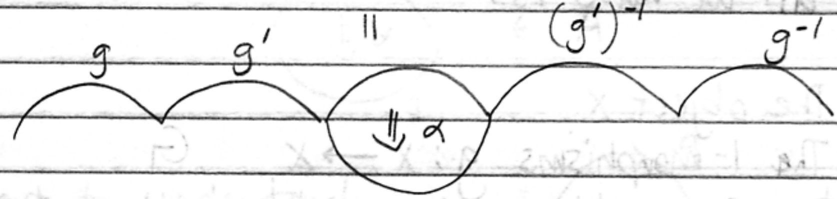

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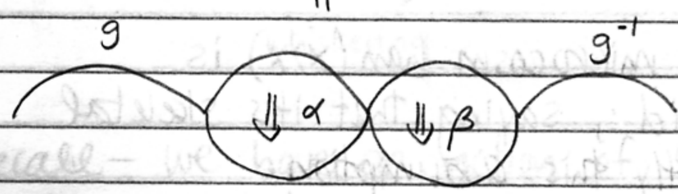
$\rho(g)\rho(g')\alpha$

② $\rho(g)(\alpha\beta)$

$\|$

$\alpha\cdot\beta$

$\|$



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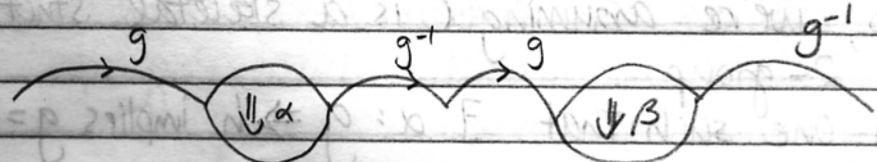
Stick in identity in between



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$(\rho(g)(\alpha))\cdot(\rho(g)(\beta))$

skeletal: only morphisms going from something to itself.

We're ultimately interested in 2-groupoids w/ only one object, x.

Our life will simplify if C is a strict 2-groupoid w/ only one object x. This is what we call a strict 2-group.

Then all we have is:

- The object x
- The 1-morphisms $g : x \to x$ G
- The 2-morphisms $\alpha : g \Rightarrow h$ — more than just A.

We can assume C is skeletal — ie) the only 2-morphisms are those going from g to g.

$\hom(x, x)$:
objects: $g : x \to x$
morphisms: $\alpha : g \Rightarrow g$

To simplify, let's assume the only 2-morphisms are of the form $\alpha : g \Rightarrow g$

Note: the microcosm $\hom(x,x)$ is a groupoid; saying that it's skeletal gives exactly this assumption.

So, we're assuming C is a skeletal strict 2-group.
ie) one such that $\exists\ \alpha : g \Rightarrow h$ implies $g = h$.

We have:

- 1 object x
- morphisms $g: x \to x$ forming a group G
- 2-morphisms $\alpha: g \Rightarrow g$
  - * if $g = \underline{1}$, these form the abelian group A.
    ↖ ident. morph.

a) We know $\overset{1}{\underset{1}{\Downarrow\alpha}} \in A$

b) Want to understand $\overset{g}{\underset{g}{\Downarrow\alpha}} \in \hom(g,g)$

We can get from a) to b) by whiskering. We have a

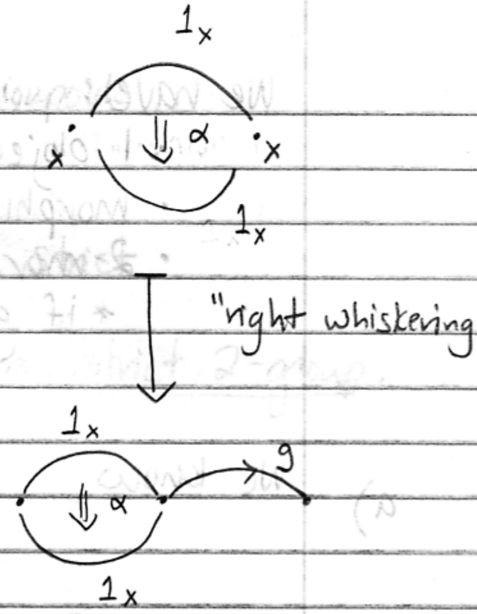
map: $A \longrightarrow \hom(g,g)$

$\xrightarrow[\text{whiskering}]{}$

Recall – we have an action of G on A by conjugation.

Define our map:

$$A \xrightarrow{f_g} \mathrm{hom}(g,g)$$



We'd like for f_g to be 1-1 & onto (ie– show it has an inverse)

f_g does have an inverse (it <u>is</u> 1-1 & onto)

f_g^{-1} = right whiskering by g^{-1}.

So– we can think of all homsets $\mathrm{hom}(g,g)$ as "being" A via f_g.

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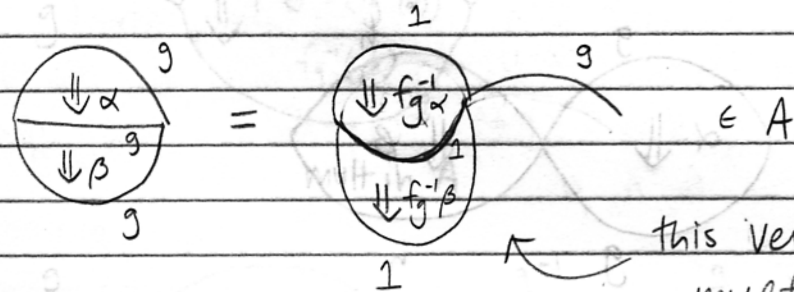
* <u>Claim</u>: The group G, the abelian group A and the action $\rho: G \to \mathrm{Aut}(A)$ determine our strict skeletal 2-group up to isomorphism.

This data clearly determines composition of 1-morphisms is like (mult. in G)
but what about vertical comp. of 2-morphisms?

in A - vertical comp is mult.

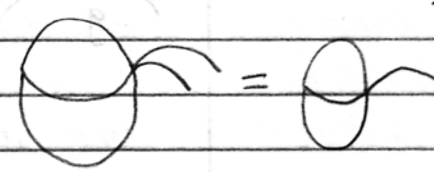


this vert. comp is mult. in A

This shows $\hom(g,g)$ w/ vertical comp. forms a group isomorphic to A via

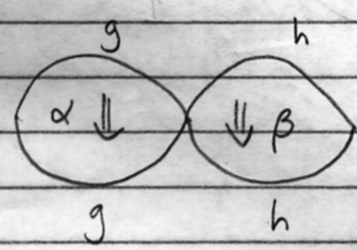
$$f_g : A \longrightarrow \hom(g,g)$$

(whisker by g then vertically comp = vertically compose then whisker by g)

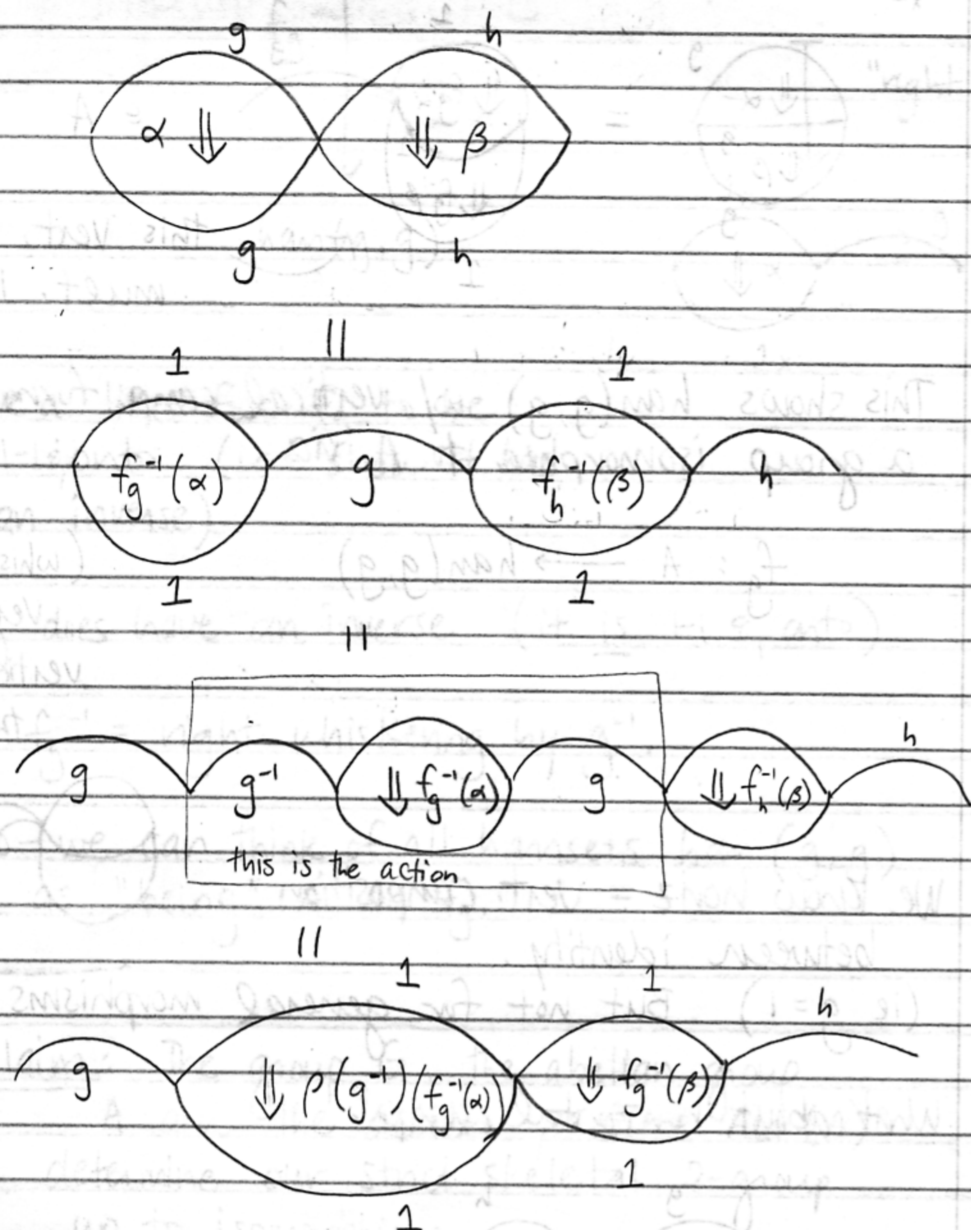


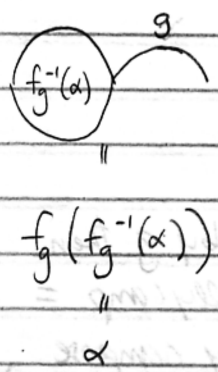
We know horiz = vert. composition between identity. (ie $g=1$) But not for general morphisms.

What about horizontal comp?



we haven't yet used our action!
we didn't use it for comp. of 1-morphisms
or in vertical comp.





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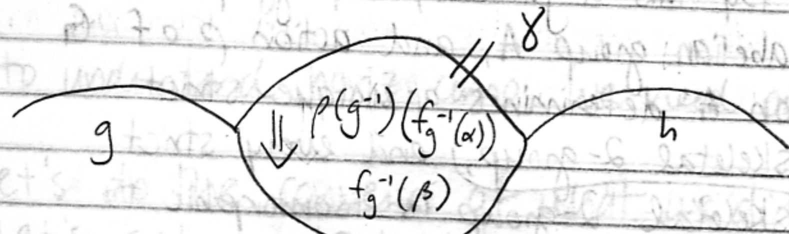
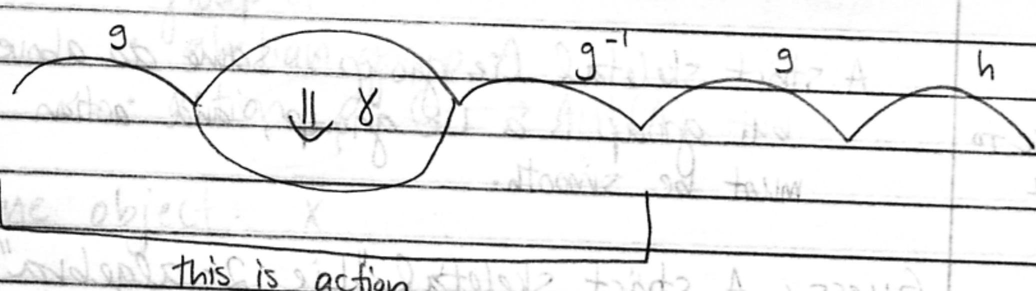


mult in A



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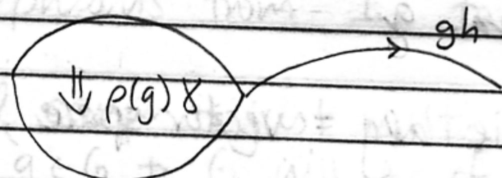

$\parallel$

$f_{gh}(\rho(g)\gamma)$

Thm: By this argument, any group G, abelian group A, and action ρ of G on A determines a unique strict skeletal 2-group, and every strict skeletal 2-group is isomorphic to such a $K(G, A, \rho)$ for unique-up-to-iso. G, A, ρ.

$K(G, A, \rho)$

lie alg = tang space to lie grp at identity

A strict skeletal Lie group: same as above but group is a Lie group, and action must be smooth.

Guess: A strict skeletal "Lie 2-algebra" should boil down to:

- a Lie algebra $\mathfrak{g}$
- an abelian Lie alg = vector space, $\mathfrak{a}$
- and a representation (action) ρ of $\mathfrak{g}$ on $\mathfrak{a}$.

"a rep of a lie alg on a v.space is a Lie 2-alg."