Continued 2-groups 1/31/02

· to understand horiz. comp, we need our action.

Let's do the converse of what we did last time:
Let's get a strict skeletal 2-group from:

- group G
- abelian group A
- action $\rho: G \longrightarrow \mathrm{Aut}(A)$

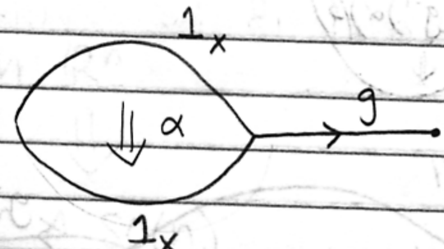
· One object: x

· {Morphisms from x to x} = G

· {2-morphisms from 1_x to itself} = A

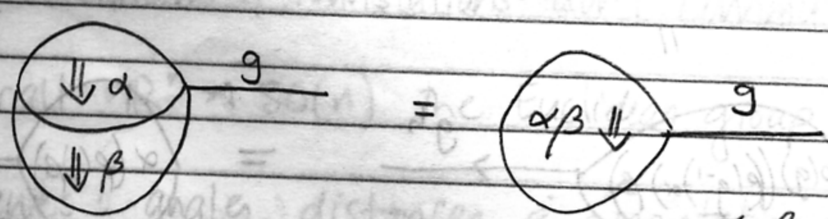
recall: skeletal- so only morphisms are bet. something & itself.

· vertical comp. is just mult. in A: any 2-morphism from $g \in G$ to G will be of the form:



$\alpha \in A$

and vertical composition goes like:



$\alpha, \beta \in A$

Recall action ρ of G on A:

$$\rho(g): \quad \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \end{tikzcd} \longmapsto \begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow\alpha"] & \bullet \arrow[r, "g^{-1}"] & \bullet \end{tikzcd}$$

- Horizontal composition also involves the action ρ of G on A:

$$\begin{tikzcd} x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha\Downarrow"] & x \arrow[r, "g"] & x \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\beta\Downarrow"] & x \arrow[r, "h"] & x \end{tikzcd} \qquad \begin{array}{l} g, h \in G \\ \alpha, \beta \in A \end{array}$$

$$\|$$

$$\begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[r, "g^{-1}"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha\Downarrow"] & \bullet \arrow[r, "g"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\beta\Downarrow"] & \bullet \arrow[r, "h"] & \bullet \end{tikzcd}$$

this is action

$$\|$$

$$\begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[r, bend left, "1_x"] \arrow[r, bend right, "1_x"'] \arrow[r, phantom, "\rho(g^{-1})\alpha\Downarrow"] & \bullet \arrow[r, bend left, "1_x"] \arrow[r, bend right, "1_x"'] \arrow[r, phantom, "\beta\Downarrow"] & \bullet \arrow[r, "h"] & \bullet \end{tikzcd}$$

do horiz. comp.

$$\|$$

$$\begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "(\rho(g^{-1})\alpha)\beta \Downarrow"] & \bullet \arrow[r, "h"] & \bullet \end{tikzcd}$$

$$\|$$

$$\begin{tikzcd} \bullet \arrow[r, "g"] & \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "(\rho(g^{-1})\alpha)\beta \Downarrow"] & \bullet \arrow[r, "g^{-1}"] & \bullet \arrow[r, "g"] & \bullet \arrow[r, "h"] & \bullet \end{tikzcd}$$

$$\|$$

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\rho(g)((\rho(g^{-1})\alpha)\beta)"] & \bullet \arrow[r, "gh"] & \bullet \end{tikzcd} = \begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\alpha(\rho(g)\beta)"] & \bullet \arrow[r, "gh"] & \bullet \end{tikzcd}$$

ρ is an action so pull g^{-1} inside to cancel

So: set of all 2-morphisms in $A \times G$ and horizontal composition is:

$$(\alpha, g) \cdot (\beta, h) = (\alpha \rho(g) \beta, gh)$$ semi-direct product

* This operation makes $A \times G$ into a group called the <u>semi-direct</u> product: $A \rtimes G$.

what's acted on (normal subgp)
A is a normal subgp of G.

'$H \triangleleft G$, normal

- groups act on their normal subgroups by conjugation.

<u>Examples of strict skeletal 2-groups</u>: preserve angles/distances

① $G = SO(n)$ - rotation group in n dimensions
$A = \mathbb{R}^n$ - translation group. (preserve angles & distances)

$SO(n)$ acts on $\mathbb{R}^n$

$\Downarrow$

$\mathbb{R}^n \rtimes SO(n)$

* note - these are lie groups, so they are lie 2-groups

<u>Note</u>: rotations & translations don't commute

We call $\mathbb{R}^n \rtimes SO(n)$ the Euclidean group

preserves: angles, distances & orientation
ie) symmetries in Euclidean geometry.

$(\alpha, 1) \in A \subseteq A \times G$. Conjugate it by something in G.

$$\underset{\underset{A\times G}{\cap}}{\underset{G}{\cap}}\hspace{-1em}(1,g)\;\underset{\underset{A\times G}{\cap}}{\underset{A}{\cap}}\hspace{-1em}(\alpha,1)\;\underset{G}{\cap}\hspace{-1em}(1,g^{-1}) = (\rho(g)\alpha, g)(1, g^{-1})$$

$$= ((\rho(g)\alpha)(\underbrace{\rho(g)1}_{=1}), gg^{-1})$$

$$= (\rho(g)\alpha, 1)$$

We also get the "Euclidean 2-group".

· again – this would be a Lie 2-group.

(2) $G = SO(n,1)$ – Lorentz group

$A = \mathbb{R}^{n+1}$ – translation grp.

$\Downarrow$

$\mathbb{R}^{n+1} \rtimes SO(n,1)$ is the Poincaré group.
We also get the Poincaré 2-group.

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(3) $G =$ any Lie group

$A = ?$ an abelian grp that G acts on.
(we could use the center of G, but the action, conjugation, would be trivial)

We let $A = \mathfrak{g}$, the Lie algebra of G.
(tang space @ identity)

G acts on $A = \mathfrak{g}$ by "adjoint rep"

$$\rho(g) X = g X g^{-1}$$ (only works for matrix groups)
if G is a matrix grp.

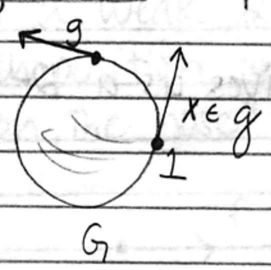
(doesn't make sense to multiply an elt from Lie alg & Lie grp)

$$\Downarrow$$

$$\mathfrak{g} \rtimes G \overset{?}{=} TG$$ the tangent bundle (a group)

We also get a 2-group.

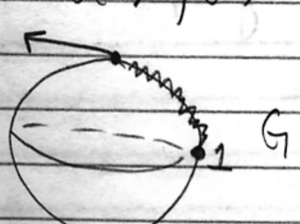
<u>tangle bundle</u> — space of all tangent vectors to manifold



How does tangent bundle become a group?

An elt in tang bundle is a pair: something in $\mathfrak{g}$, G.
think of elt in tang bundle as equiv. classes of paths.

$\gamma(t)\,\eta(t)$ mult of paths, mod out by equiv operation



$$\{\text{paths from } 1 \in G \text{ to some pt } g \in G\}$$
$$\downarrow$$
$$TG$$

<u>Claim</u>: if G is the Lorentz group $SO(3,1)$
this 2-group plays a role in General Relativity.

This should allow us to think of GR as a categorified gauge theory w/ "Lorentz 2-group" as its "gauge grp."

There's a nice way to come up w/ a 2 group from the quaternions.

The quaternions $\mathbb{H}$ are a 4d algebra (normed division alg: $\|xy\| = \|x\|\|y\|$)

The unit quaternions form a group: $S^3 \cong SU(2)$ which acts on $\mathbb{H}$ via

$$g: x \longmapsto gxg^{-1}.$$

So — $SU(2)$ acting on $\mathbb{H}$ gives us a 2-group.

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strict vs. skeletal

everything holds on the nose

if there's a morphism between 2 things, those things must be the same

We have categories of groups, etc.

So we have a 2-category of 2-groups.

Note: we have a 2-category & 3-category of 2-groups.

Thm: Every 2-group is equivalent to a strict 2-group. (There is a 2-category of 2-groups which we are implicitly using here.)

Thm: Every 2-group is equivalent to a skeletal 2-group.

Anti-Thm: Not every 2-group is equiv. to a strict skeletal one.

We've talked about strict 2-categories and defined a strict 2-group in terms of those. How about general weak 2-categories?

In a weak 2-category, associativity of composition of 1-morphisms doesn't hold "on the nose": Instead we have an associator:

$$\bullet \xrightarrow{f} \bullet \xrightarrow{g} \bullet \xrightarrow{h}$$

$$A_{f,g,h} : (fg)h \overset{\sim}{\Rightarrow} f(gh)$$

$A_{f,g,h}$ satisfies a bunch of laws "on the nose" because it's a two-morphism and we're in a 2-category. (weak)

For example, satisfies pentagon id:

$$\begin{tikzcd}[column sep=small]
& (fg)(hk) \arrow[dr, Rightarrow, "A_{f,g,hk}"] & \\
((fg)h)k \arrow[ur, Rightarrow, "A_{fg,h,k}"] \arrow[d, Rightarrow, "A_{f,g,h}\cdot k"'] & & f(g(hk)) \\
(f(gh))k \arrow[rr, Rightarrow, "A_{f,gh,k}"'] & & f((gh)k) \arrow[u, Rightarrow, "f\cdot A_{g,h,k}"']
\end{tikzcd}$$

whiskering

Also have left/right unit laws: (they're 2 morphisms)

$$l_x : 1_x f \overset{\sim}{\Rightarrow} f \qquad \forall\, f : x \to y$$

$$r_x : f 1_x \overset{\sim}{\Rightarrow} f \qquad \forall\, f : y \to x$$

satisfying

whiskering

whiskering

$$\begin{tikzcd}
f1_y \arrow[drr, Rightarrow, "r_y"'] & (1_x f)1_y \arrow[l, Rightarrow, "l_x \cdot 1_y"'] \arrow[r, Rightarrow, "A_{1_x,f,1_y}"] & 1_x(f1_y) \arrow[r, Rightarrow, "1_x\cdot r_y"] & 1_x f \arrow[dl, Rightarrow, "l_x"] \\
& & f &
\end{tikzcd}$$

* The above 2 laws are enough to define a weak 2-category *

Secret Thm: (Never seen proved before)

Every weak 2-category is equivalent to a skeletal one where l_x, r_y are identities.

Using this theorem, we expect that a weak skeletal 2-group will involve:

- a group G
- an abelian group A
- an action $\rho: G \longrightarrow Aut(A)$
- A map $\alpha: G^3 \longrightarrow A$ satisfying pentagon id.

maybe something else? In 2nd diagram – we'd need $A_{1_x, f, 1_y}$ to be id_f.

called a 3-cocycle on G w/ values in A.

"group cohomology" (alg. topology)