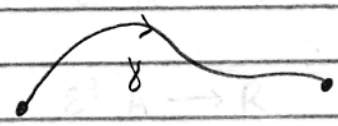
2/12/02 **Electrostatics**

To understand "categorified gauge theory" we need to understand "gauge theory" (which describes nature). To understand that, we need to start with electromagnetism.



gives $e^{i\int_\gamma A} \in U(1)$

Let's go back – to electrostatics.

Electrostatics:

$$\begin{cases} \cdot\ \text{electric field} & \vec{E} : \underset{\text{space}}{\mathbb{R}^3} \longrightarrow \mathbb{R}^3 \\ \cdot\ \text{charge density} & \rho : \mathbb{R}^3 \longrightarrow \mathbb{R} \end{cases}$$

(lots of current, field pts outward)

$\vec{\nabla} \cdot \vec{E} = \rho$ divergence

$\vec{\nabla} \times \vec{E} = 0$ curl

since curl of gradient = 0, we can use this

trick: if we let $\vec{E}$ be the gradient of a function:

$\vec{E} = -\vec{\nabla}\phi$ for some funct. $\phi : \mathbb{R}^3 \to \mathbb{R}$

the electrostatic potential, then our 2nd eqn

$$\vec{\nabla} \times \vec{E} = 0$$

is automatically satisfied (and all solns are of this form), so the only interesting equation is:

$$\boxed{\nabla^2 \phi = -\rho}$$ Poisson's eqn.

Magnetostatics

- magnetic field $\vec{B} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

- Current density $\vec{j} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$

$$\vec{\nabla} \cdot \vec{B} = 0$$ (divergence) ie - there are no such thing as magnetic changes

$$\vec{\nabla} \times \vec{B} = \vec{j}$$

The trick: $\operatorname{div}(\operatorname{curl} F) = 0$ always

so, we let

$$\vec{B} = \vec{\nabla} \times \vec{A}$$ (the curl of something)

for some vector field $\vec{A} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ the "vector potential"

then

$$\vec{\nabla} \cdot \vec{B} = 0$$ is automatic (and all solutions are of this form)

so the only interesting eqn is:

$$\boxed{\vec{\nabla} \times (\vec{\nabla} \times \vec{A}) = \vec{j}}$$

Electromagnetism:

Now we make things be functions of time, too:

$$\vec{E}: \mathbb{R}^4 \longrightarrow \mathbb{R}^3 \quad ; \quad \vec{E}(t,x,y,z) \in \mathbb{R}^3$$

$$\vec{B}: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

$$\rho: \mathbb{R}^4 \longrightarrow \mathbb{R}$$

$$\vec{j}: \mathbb{R}^4 \longrightarrow \mathbb{R}^3$$

Now — electricity and magnetism get unified via Maxwell's eqns:

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \rho & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t} & \vec{\nabla}\times\vec{B} = \vec{j} + \dfrac{\partial \vec{E}}{\partial t} \end{array}}$$

this term is maxwell's idea

These eqns are related to conservation of charge:

Conservation of charge:

$$\frac{d\rho}{dt} = \frac{d}{dt}\vec{\nabla}\cdot\vec{E}$$

$$= \vec{\nabla}\cdot\frac{\partial \vec{E}}{\partial t}$$

$$= \vec{\nabla}\cdot\left(\vec{\nabla}\times\vec{B} - \vec{j}\right)$$

$$= -\vec{\nabla}\cdot\vec{j} \qquad \text{since div curl } \vec{B} = 0.$$

this eqn needs Maxwell's term in order to hold.

Also - wave solutions give light.

The trick for solving 2 of Maxwell's eqns:

$$\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$$

$$\vec{B} = \vec{\nabla} \times \vec{A}$$

To show eqns in box hold:

Now - $\vec{\nabla}\cdot\vec{B} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{A}) = 0$ ✓

as before, but also

$$\vec{\nabla}\times\vec{E} = -\vec{\nabla}\times\left(\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}\right)$$

$$= -\vec{\nabla}\times\frac{\partial \vec{A}}{\partial t}$$

$$= -\frac{\partial}{\partial t}\vec{\nabla}\times\vec{A}$$

$$= -\frac{\partial}{\partial t}\vec{B} \checkmark$$

This gives all solutions.

So the source-free equations (eqns w/out $\rho, \vec{j}$) follow automatically if we write $\vec{E}$ and $\vec{B}$ using potentials; the other 2 become 2nd order eqns in $\phi, \vec{A}$.

we want to rewrite all of these eqns in the language of differential forms.

Differential Forms

What's "df" that we see in Calculus?
ie– dx, du ← substitution
under integrals

Let's take the algebra of smooth functions $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ (or $f: M \longrightarrow \mathbb{R}$ for any smooth manifold) and throw in things like df (where f is a smooth function) to get a bigger algebra satisfying these rules: (in addition to the rules of being an algebra)

$$d(f+g) = df + dg$$

$$d(fg) = (df)g + f(dg)$$

$$d(\alpha f) = \alpha\, df \qquad \alpha \in \mathbb{R}$$

It now follows that $d\alpha = 0$.

$$df\, dg = -dg\, df$$

This is why there isn't $\iint dx\, dx$

since $dx\, dx = -dx\, dx$ so this is zero.

Many rules from multivariable calculus follow!
(like when making substitutions we get $\left|\begin{matrix}\frac{\partial f}{\partial x} & \frac{\partial f}{\partial x} \\ \vdots & \vdots\end{matrix}\right|$) 10B.

It follows that:

- $d\alpha = 0$ and
- $df = \frac{\partial f}{\partial x^1} dx^1 + \dots + \frac{\partial f}{\partial x^n} x^n$

notice - this is like the gradient of f. (keeps track of partials)

ie) df is like $\vec{\nabla} f$.

We call this big algebra the differential forms on $\mathbb{R}^n$, $\Omega(\mathbb{R}^n)$. Every differential form is a $\mathbb{R}$-linear combination of p-forms

$$f\, dg_1\, dg_2 \dots dg_p \qquad 1 \le p \le n$$

We let $\Omega^p(\mathbb{R}^n)$, the p-forms be the space of linear combs. of things like

$$f dg_1 \cdots dg_p \quad \text{where } p \text{ is fixed.}$$

Ex) In $\mathbb{R}^3$ we have

$$\Omega^0(\mathbb{R}^3) = \text{zero forms} = \{f : \mathbb{R}^3 \longrightarrow \mathbb{R}\}$$
(no dg_p's)

Note: electric field – 1 form
magnetic field – 2 form

1-forms $\Omega^1(\mathbb{R}^3) = \{ f_1 dx + f_2 dy + f_3 dz \mid f_i : \mathbb{R}^3 \to \mathbb{R} \}$

2-forms $\Omega^2(\mathbb{R}^3) = \{ f_{12} dx dy + f_{23} dy dz + f_{13} dx dz \mid f_{ij} : \mathbb{R}^3 \to \mathbb{R} \}$

3-forms $\Omega^3(\mathbb{R}^3) = \{ f dx dy dz \mid f : \mathbb{R}^3 \to \mathbb{R} \}$

We can define

derivative/ differential $\quad d : \Omega^p(\mathbb{R}^n) \longrightarrow \Omega^{p+1}(\mathbb{R}^n)$

by

$$d(f dg_1 \cdots dg_p) = df\, dg_1 \cdots dg_p$$ and d is linear.

This is the unique "d" that satisfies:

$$d(u + w) = du + dw \qquad u, w \in \Omega^p(\mathbb{R}^n)$$

$$d(uw) = (du)w + (-1)^p u\, dw$$

$u \in \Omega^p(\mathbb{R}^n)$

$$d(f) = df$$

$w \in \Omega^q(\mathbb{R}^n)$

old df $\qquad$ so $uw \in \Omega^{p+q}(\mathbb{R}^n)$

$$\left.\begin{array}{l} d(\text{0-form}) = \text{gradient} \\ d(\text{1-form}) = \text{curl} \\ d(\text{2-form}) = \text{divergence} \end{array}\right\} \text{in } \mathbb{R}^3$$