2/14/02

Correction: $\Omega(\mathbb{R}^n)$ is the associative algebra generated by f, df ($f: \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth) satisfying relations:

$$d(f+g) = df + dg$$

$$d(\alpha f) = \alpha df \qquad \alpha \in \mathbb{R}$$

$$d(fg) = (df)g + fdg$$

this one was left out

$$fdg = dgf$$

$$dfdg = -dgdf$$

Example: why is $r\,dr\,d\theta = dx\,dy$? (when changing to polar)

$$x = r\cos\theta \qquad y = r\sin\theta$$

$$dx = d(r\cos\theta) = dr\cos\theta + r\,d(\cos\theta)$$

Thm: For all $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ smooth, we have

$$df = \frac{\partial f}{\partial x_1} dx_1 + \dots + \frac{\partial f}{\partial x_n} dx_n$$

using this thm we get

Note: $drdr = -drdr$

$$= (dr)\cos\theta - r\sin\theta\, d\theta$$

Similarly – $dy = dr\sin\theta + r\cos\theta\, d\theta$

$$dxdy = (dr)r\cos^2\theta\, d\theta - r\sin^2\theta\, drd\theta$$

$$= r\,dr\,d\theta$$

Rule: $\mu w = (-1)^{pq} w \mu$
where $\mu \in \Omega^p \quad w \in \Omega^q$

There exists a unique linear operator

$$d : \Omega(\mathbb{R}^n) \longrightarrow \Omega(\mathbb{R}^n)$$

st $d(f\,dg_1 \cdots dg_p) = df\,dg_1 \cdots dg_p$

We call d the exterior derivative.

Thm: $d^2 = 0$
ie) $\forall\, w \in \Omega(\mathbb{R}^n)$ we have
$$ddw = 0$$

secret one!

proof: $dd(f\,dg_1 \ldots dg_p) = d(1\,df\,dg_1 \ldots dg_p)$

$= d1\,df\,dg_1 \ldots dg_p$

$= 0$ since $d\alpha = 0$ for constant α.

$d\underline{1} = 0$ since $d1 = d11 = (d1)\underline{1} + 1d1$
(product rule)

$= d\underline{1} + d1$

so $d\underline{1} = d1 + d1 \Rightarrow d1 = 0.$

functions $\Omega^0(\mathbb{R}^3) = \{\text{smooth functs } f\}$

$\boxed{\text{GRAD}}$ $\downarrow d$

1-forms $\Omega^1(\mathbb{R}^3) = \{f_1\,dx_1 + f_2\,dx_2 + f_3\,dx_3\}$ f_i smooth

$\boxed{\text{CURL}}$ $\downarrow d$

2-forms $\Omega^2(\mathbb{R}^3) = \{f_1\,dx_2dx_3 + f_2\,dx_3dx_1 + f_3\,dx_1dx_2\}$

$\boxed{\text{DIV}}$ $\downarrow d$ Note - no dx_1dx_1 since $dx_1dx_1 = -dx_1dx_1 \Rightarrow dx_1dx_1 = 0$

3-forms $\Omega^3(\mathbb{R}^3) = \{f\,dx_1dx_2dx_3\}$

$\downarrow d$

4-forms $\Omega^4(\mathbb{R}^3) = \{0\}$

(d takes a p form & produces $(p+1)$ form)

For example:

$$d(f\,dx_1) = df\,dx_1 \qquad \text{this is curl!}$$

$$= \left(\frac{\partial f}{\partial x_1}dx_1 + \frac{\partial f}{\partial x_2}dx_2 + \frac{\partial f}{\partial x_3}dx_3\right)dx_1$$

$$= -\frac{\partial f}{\partial x_2}dx_1dx_2 + \frac{\partial f}{\partial x_3}dx_3dx_1$$

$$\nabla \times (f,0,0) = \left(0, \frac{\partial f}{\partial x_3}, -\frac{\partial f}{\partial x_2}\right)$$

So - $d: \Omega^1(\mathbb{R}^3) \longrightarrow \Omega^2(\mathbb{R}^3)$ is the curl.

$$\nabla \times (f,0,0) = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f & 0 & 0 \end{vmatrix} = -j\left(0 - \frac{\partial f}{\partial x_3}\right) + k\left(0 - \frac{\partial f}{\partial x_2}\right)$$

Similarly - $d : \Omega^2(\mathbb{R}^3) \longrightarrow \Omega^3(\mathbb{R}^3)$
and

$d^2 = 0$ gives

$\nabla \times (\nabla \phi) = 0$ curl of grad $= 0$

$\nabla \cdot (\nabla \times \vec{A}) = 0$ div of curl $= 0$

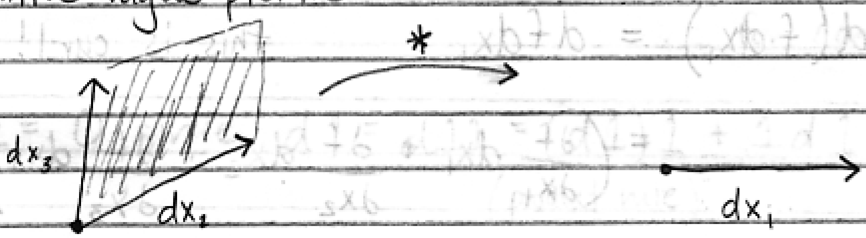
There is also an operator:

$$* : \Omega^p(\mathbb{R}^n) \longrightarrow \Omega^{n-p}(\mathbb{R}^n)$$

called the Hodge-star operator, which depends on the metric (dot product) and orientation on $\mathbb{R}^n$.

Intuitive vague picture:

(cheating - these forms are co-tang vectors)

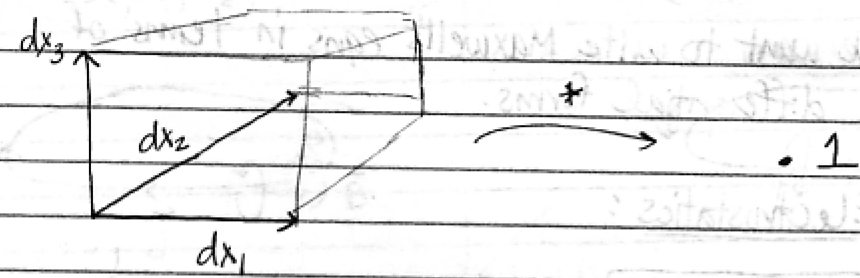


· then $dx_2 dx_3$ is parallelogram

$$(dx_2 dx_3) dx_1 = \underbrace{dx_1 dx_2 dx_3}_{\substack{\text{Volume form} \\ \text{(oriented basis)}}}$$

$*$ operator spits out something $\perp$ to input.

Ex)



$$dx_1 dx_2 dx_3 \qquad\qquad dx_1 dx_2 dx_3\ 1 = dx_1 dx_2 dx_3$$

In Euclidean $\mathbb{R}^n$ (or any oriented Riemannian n-manifold) we have

$$**\omega = (-1)^{p(n-p)}\,\omega$$

where ω is a p-form. $(\omega \in \Omega^p(\mathbb{R}^n))$

Note: In $\mathbb{R}^3$ – this sign goes away $(n=3,$ so $p=0,1,2,3)$

$$\begin{array}{ll}
 & \Omega^0(\mathbb{R}^3) \\
\boxed{\text{DIV}} & \uparrow *d* \\
 & \Omega^1(\mathbb{R}^3) \\
\boxed{\text{CURL}} & \uparrow *d* \\
 & \Omega^2(\mathbb{R}^3) \\
\boxed{\text{GRAD}} & \uparrow *d* \\
 & \Omega^3(\mathbb{R}^3)
\end{array}$$

We want to write Maxwell's eqns in terms of differential forms.

Electrostatics:

We can think of $\vec{E}$ as a 1-form or 2-form,

$$\boxed{\begin{aligned} \vec{\nabla} \times \vec{E} &= 0 \\ \vec{\nabla} \cdot \vec{E} &= \rho \end{aligned}}$$ (implied by $\vec{E} = -\vec{\nabla}\phi$)

We'll think of $\vec{E}$ as a 1-form on $\mathbb{R}^3$, called E.
We'll think of ϕ as a zero form (or function)
ρ as a zero form

Then, we rewrite our 2 above eqns to get

$$\boxed{\begin{aligned} dE &= 0 \\ *d*E &= \rho \end{aligned}}$$ *implied by $E = -d\phi$ and $d^2 = 0$.

Magnetostatics:

$$\boxed{\begin{aligned} \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{B} &= \vec{j} \end{aligned}}$$ (implied by $\vec{B} = \vec{\nabla} \times \vec{A}$)

Think of $\vec{B}$ as a 2-form on $\mathbb{R}^3$, B
$\vec{A}$ as a 1-form, A
$\vec{j}$ as a 1-form, j

$$\boxed{\begin{aligned} dB &= 0 \\ *d*B &= j \end{aligned}}$$ implied by $B = dA$ and $d^2 = 0$.

$(B_x, B_y, B_z) \longmapsto (B_x, -B_y, -B_z)$

$x \to -x$

$y \to -y$

$z \to z$

consistent w/ $B = B_x\,dy\,dz + B_y\,dz\,dx + B_z\,dx\,dy$

$\vec{j}$ is a vector; j is a 1-form

$\vec{B}$ is a pseudovector; B is a 2-form

<u>Electromagnetism</u>:

$* \left[\; \vec{\nabla} \cdot \vec{B} = 0 \qquad \vec{\nabla} \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \right.$

$*** \left(\text{follows from } \vec{B} = \vec{\nabla} \times \vec{A} \text{ and } \vec{E} = -\nabla\phi - \dfrac{\partial \vec{A}}{\partial t} \right)$

$** \left[\; \vec{\nabla} \cdot \vec{E} = \rho \qquad \vec{\nabla} \times \vec{B} - \dfrac{\partial \vec{E}}{\partial t} = \vec{j} \right.$

Let $F = \underbrace{E\,dt}_{\text{1-form on } \mathbb{R}^4} + \underbrace{B}_{\text{2 form on } \mathbb{R}^4}$

be a 2-form on $\mathbb{R}^4$ – electromagnetism.

$a = \phi\,dt + A$ be a 1-form on $\mathbb{R}^4$

$J = \rho\,dt + j$ be a 1-form on $\mathbb{R}^4$

Doing this, we get the top 2 eqns to collaspe to one eqn, and bottom 2 eqns to do the same.

(*) boils down to $dF = 0$ (trivial Maxwell eqn)
(follows from $\underbrace{F = dA}_{***}$) and $d^2A = 0$.

(**) boils down to $*d*F = J$

defined on Minkowski metric

So– if we work using the 4-potential A, all of electromagnetism boils down to

$\boxed{\text{Maxwell's eqn}}$: $\underbrace{*d*d}_{\text{space-time operator similar to "Laplacian"}} A = J$ (2^{nd} order D.E. since 2 derivatives)

explains electricity, magnetism & light