Note — · gauge grp of Electromagnetism is $U(1)$

· categorification: boost up a dimension

2/19/02

$$\text{Space} = \mathbb{R}^3 \ni (x,y,z) = (x_1, x_2, x_3)$$

$$\text{Spacetime} = \mathbb{R}^4 \ni (t,x,y,z) = (x_0, x_1, x_2, x_3)$$

Electromagnetism

electric field

E time-dependent "$\vec{E}$"
1-form on space

B time-dependent "$\vec{B}$"
2-form on space
magnetic field

$F = E\,dt + B$
2-form on spacetime

If $E = E_i\,dx^i$ $(i,j = 1,2,3)$
Einstein sum convention

$B = \frac{1}{2}\varepsilon_{ijk} B^i dx^j dx^k$

then

$$F = \underbrace{E_i\,dx^i\,dt}_{\text{"time/space"}} + \underbrace{\tfrac{1}{2}\varepsilon_{ijk} B^i dx^j dx^k}_{\text{"space/space"}}$$

or if

$F = \frac{1}{2} F_{\alpha\beta}\, dx^\alpha dx^\beta \quad \alpha,\beta = 0,1,2,3$

Categorified Electromagnetism

E time-dependent "pseudo v. field"
2-form on space "$\vec{E}$"

B time-dependent "b"
3-form on space "pseudo scalar field"

$F = E\,dt + B$
3-form on spacetime

$E = \frac{1}{2}\varepsilon_{ijk} E^i dx^j dx^k$

$B = b\,dx\,dy\,dz$ then

$F = \frac{1}{2}\varepsilon_{ijk} E^i dx^j dx^k dt + b\,dx\,dy\,dz$

or

$F = \frac{1}{6} F_{\alpha\beta\gamma}\, dx^\alpha dx^\beta dx^\gamma$

Last time: functs / fields $\Omega(\mathbb{R})$

$$\Big\downarrow d \quad \text{grad}$$

v. fields $\Omega^1(\mathbb{R})$

$$\Big\downarrow d \quad \text{curl}$$

(pseudo) v. fields $\Omega^2(\mathbb{R})$

$$\Big\downarrow d \quad \text{div}$$

(pseudo) functs / scalar fields $\Omega^3(\mathbb{R})$

cont'd LHS list

Electromagnetism

then

$$F_{\alpha\beta} = \begin{array}{c} \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & B_z & -B_y \\ E_y & -B_z & 0 & B_x \\ E_z & B_y & -B_x & 0 \end{pmatrix} \begin{array}{l} \text{] time} \\ \\ \text{] space} \\ \\ \end{array} \\ \underbrace{\quad}_{\text{time}} \qquad \underbrace{\qquad\qquad\qquad}_{\text{space}} \end{array}$$

electromagnetic field!

Electromagnetism

Sourcefree Maxwell's eqn.
(don't involve matter or metric)

$dF = 0$ iff $d(E\,dt + B) = 0$

iff $dE\,dt - E\,d\,dt + dB = 0$

$\quad\quad\quad\quad\quad \overset{\shortparallel}{0}$ since $d^2 = 0$

see Note next pg

iff $d_s E\, dt + dt \dfrac{\partial E}{\partial t}\, dt \to 0$

$+ d_s B + dt \dfrac{\partial B}{\partial t} = 0$

$\quad\quad \underbrace{}_{(*)}$

iff $\left(d_s E + \dfrac{\partial B}{\partial t}\right) dt + d_s B = 0$

(*) here we switched a 1-form & 2-form, so we get 2 minus signs.

iff $d_s E + \dfrac{\partial B}{\partial t} = 0$ & $d_s B = 0$

or: $\vec{\nabla} \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0$ & $\nabla \cdot \vec{B} = 0$

Categorified Electromagnetism

Sourcefree 2-Maxwell eqn.

$dF = 0$ iff ... same argument holds ...

iff $d_s E\,dt + d_s B + dt \dfrac{\partial B}{\partial t} = 0$

iff $\left(d_s E - \dfrac{\partial B}{\partial t}\right) dt = 0$ and

$d_s B = 0$

or: $\vec{\nabla} \cdot E - \dfrac{\partial b}{\partial t} = 0$

$d_s B = 0$

$\nearrow\ 0 = 0$

B is a 3-form, d of it a 4-form so zero.

Note: d on $\mathbb{R}^4$ (spacetime) is related to d on $\mathbb{R}^3$ (space) ("d_s") as follows:

$$d\omega = d_s\omega + dt\,\frac{\partial\omega}{\partial t}$$

say $\omega = f\,dx_{\alpha_1}\dots dx_{\alpha_k}$

$$d\omega = df\,dx_{\alpha_1}\dots dx_{\alpha_k}$$

$$= \frac{\partial f}{\partial x_\alpha}\,dx_\alpha\,dx_{\alpha_1}\dots dx_{\alpha_k}$$

$i=1,2,3$
$$= \underbrace{\frac{\partial f}{\partial x_i}\,dx_i\,dx_{\alpha_1}\dots dx_{\alpha_k}}_{\text{space}} + \underbrace{\frac{\partial f}{\partial t}\,dt\,dx_{\alpha_1}\dots dx_{\alpha_k}}_{\text{time}}$$

$$= d_s\omega + dt\,\frac{\partial\omega}{\partial t}$$

Electromagnetism $E = -\nabla\phi$

The potentials:

ϕ is a 0-form on space "ϕ"

A is a 1-form on space "$\vec{A}$"

$a = -\phi\, dt + A$

If $F = da$ then the boring Maxwell eqn $dF = 0$ follows from $d^2 = 0$.

$F = da$ iff $\underbrace{E dt} + B = \underbrace{dt \frac{da}{dt}} + d_s a$

space parts = space parts
time parts = time parts

$$E dt + B = dt \frac{d}{dt}(-\phi\, \cancel{dt} + A) + d_s(-\phi\, dt + A)$$

$dt\, dt = 0$

$$= \left(-\frac{\partial A}{dt} - d_s\phi\right) dt + d_s A$$

this is true iff

$$E = -\frac{\partial A}{dt} - d_s\phi$$

$$B = d_s A \qquad \text{or}$$

$$\vec{E} = -\frac{\partial \vec{A}}{dt} - \vec{\nabla}\phi \quad \&, \quad \vec{B} = \vec{\nabla}\times\vec{A}$$

Categorified Electromagnetism

The potentials:

ϕ is a 1-form on space "$\vec{\phi}$"

A is a 2-form on space "$\vec{A}$"

$a = -\phi\, dt + A$

If $F = da$ then $dF = 0$ follows from $d^2 = 0$.

$F = da$ iff ...

same argument works up until:

$$E dt + B = \left(\frac{\partial A}{dt} - d_s\phi\right) dt + d_s A$$

iff

$$E = \frac{\partial A}{dt} - d_s\phi \quad \text{and}$$

$$B = d_s A$$

or

$$\vec{E} = \frac{\partial \vec{A}}{dt} - \vec{\nabla}\times\vec{\phi}$$

$$b = \vec{\nabla}\cdot\vec{A}$$

Now - talk about eqns that do involve matter
(Hodge * operator)

Hodge - * operator

If V is a real vector space then we can form

$$T^pV = \underbrace{V \otimes \dots \otimes V}_{p \text{ times}}$$

The permutation group S_p acts on T^pV

$$\sigma(v_1 \otimes \dots \otimes v_p) = v_{\sigma^{-1}(1)} \otimes \dots \otimes v_{\sigma^{-1}(p)}$$

Define the symmetric tensors:

$$S^pV = \{ x \in T^pV \mid \sigma x = x \}$$ don't change

and

antisymmetric tensors:

$$\Lambda^pV = \{ x \in T^pV \mid \sigma x = (-1)^{\operatorname{sgn}\sigma} x \}$$

A p-form on V is the same as a smooth function

$$\omega : V \longrightarrow \Lambda^pV^*.$$

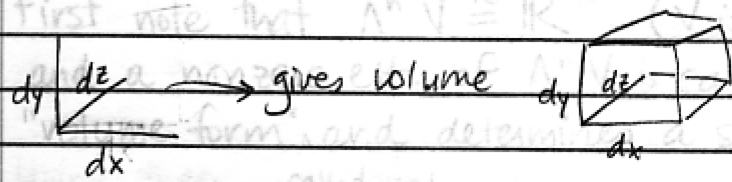
If $e_1, \ldots, e_n$ is a basis of V then let

$dx_1, \ldots, dx_n$ be the dual basis, and then ω can be written as

(sum over $i_1 \ldots i_p$) $\underbrace{\omega_{i_1 \ldots i_p}}_{\mathbb{R}\text{-valued functs on } V} \; \underbrace{dx^{i_1} \ldots dx^{i_p}}_{\text{is a basis of } \Lambda^p V^*}$

exterior

If V is n-dim'l, then $\Lambda^n V$ is 1-dim'l w/ basis $\underbrace{dx_1 \ldots dx_n}_{\text{called a volume form}}$



Any nonzero elt. of $\Lambda^n V$, or "volume form" will be a basis of $\Lambda^n V$. Given any volume form, say vol, we get an isomorphism:

or: $\Lambda^p V^* \cong \Lambda^{n-p} V$

$\Lambda^p V \cong \Lambda^{n-p} V^*$ defined as below:

(recall — Hodge-star operator takes a p-form to an $(n-p)$ form.)

multiplication: $\Lambda^p V \times \Lambda^{n-p} V \longrightarrow \Lambda^n V \cong \mathbb{R}$

$$\mathrm{vol} \longmapsto 1$$

gives

$$\Lambda^p V \longrightarrow (\Lambda^{n-p} V)^* \cong \Lambda^{n-p} V^*$$

god-given

which is an isomorphism.

We eventually want to talk about

$*d*F = J$ (non-trivial Maxwell eqn)