2/21/02

Maxwell's eqns that do involve matter (involve Hodge-star operator)

V : n-dim'e v.space

"exterior alg" $\Lambda V = \bigoplus_{p=0}^{\infty} \Lambda^p V$ = algebra generated by V w/ relations $vw = -wv$

{note- symmetric alg is alg. gen by V w/ $vw = wv$}

Differential forms on V, $\Omega(V)$, are the same as smooth functions $\omega : V \longrightarrow \Lambda(V^*)$

First note that $\Lambda^n V \cong \mathbb{R}$ (V is a real v.space) and a nonzero elt. of $\Lambda^n V$ is called a "volume form", called vol. and determines a specific isomorphism

$$\begin{aligned} \Lambda^n V &\longrightarrow \mathbb{R} \\ \text{vol} &\longmapsto 1 \end{aligned}$$

Then multiplication

$$\text{mult}: \Lambda^{n-p} V \otimes \Lambda^p V \longrightarrow \Lambda^n V \xrightarrow{\sim} \mathbb{R}$$

defines a linear map

$$\Lambda^p V \longrightarrow (\Lambda^{n-p} V)^* \cong \Lambda^{n-p} V^*$$

god-given iso.

this isn't exactly the Hodge-star operator, but similar nondegenerate means we get an iso $V \cong V^*$

If we have a metric

$g: V \otimes V \longrightarrow \mathbb{R}$ (symm, nondegenerate) on V

e.g. $\mathbb{R}^3$ w/ $g(e_i, e_j) = g_{ij}$ $\quad \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$i, j = \underbrace{1, 2, 3}_{\text{space}}$

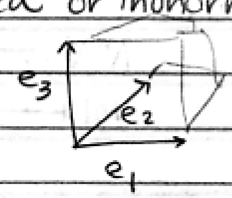
in $\mathbb{R}^4$ w/ $g(e_\alpha, e_\beta) = e_{\alpha\beta}$

$\alpha, \beta = \underset{\text{time}}{0}, \underbrace{1, 2, 3}_{\text{space}}$ $\quad \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$

and an orientation, we get a volume form:

$$e_1 e_2 e_3 \ldots e_n \in \Lambda^n V$$

where $e_1, \ldots, e_n$ is any oriented orthonormal basis of V, giving us our map



$$\Lambda^p V \longrightarrow \Lambda^{n-p} V^*$$

multiply these oriented orthonormal basis elts, get a vol. form (n-cube)

but the metric gives an iso. $V \cong V^*$, so we get the Hodge-$*$ operator:

$$*: \Lambda^p V \longrightarrow \Lambda^{n-p} V \qquad \text{or}$$

$$*: \Lambda^p V^* \longrightarrow \Lambda^{n-p} V^*$$

Ex) in 3-dimensions ($\mathbb{R}^3$)

$*: dx \longrightarrow dydz$

$*: dydz \longrightarrow dx$

Ex) in $\mathbb{R}^4$ — it's one of these!

$*: dt \longmapsto \pm\, dxdydz$ (1-form) $\longmapsto$ (3-form)

$*: dxdy \longmapsto \pm\, dtdz$

$*: dtdx \longmapsto -\, dydz$

If $V = \mathbb{R}^3$ we get Hodge-star operator on space

$$*_s : \Lambda^p V \longrightarrow \Lambda^{3-p} V$$

If $V = \mathbb{R}^4$ we get (using Minkowski metric)

$$* : \Lambda^p V \longrightarrow \Lambda^{4-p} V$$

Using these we define $*$ on diff. forms on space/spacetime. If ω is a p-form on spacetime we can write:

p-form $\omega = dt\,\underbrace{\omega_0}_{\text{time dependent } (p-1) \text{ form on space}} + \omega_s$ time dependent p-form on space

$$*\,\omega = -\,*_s\,\omega_0 + (-1)^p\,dt\,*_s\,\omega_s$$

ex) $*(dtdx) = -*_s dx = -dydz$ by example above funcs / scalars $\Omega^0(\mathbb{R}^3)$

$\uparrow$ DIV $*_s d_s *_s$

vectors $\Omega^1(\mathbb{R}^3)$

$\uparrow$ CURL $*_s d_s *_s$

pseudo vectors $\Omega^2(\mathbb{R}^3)$

$\uparrow$ GRAD $*_s d_s *_s$

pseudo scalars $\Omega^3(\mathbb{R}^3)$

Electromagnetism

ρ, the charge density is time-dependent 0-form on space, "ρ"

j, the current, is time-dependent 1-form on space, "$\vec{j}$" ("vector")

$J = -\rho\, dt + j$ 1-form on spacetime

The interesting Maxwell eqn. says

$$*d*F = J$$

(we'll break this up into space & time parts)

each side is a 1-form

This gives 2 eqns when we write

$$F = E\,dt + B$$

2-form, $p=2$; E: 1-form on space; B: 2-form on space

$$= -dt\,E + B$$

since switch a 1-form

$$\omega\mu = (-1)^{pq}\mu\omega$$

ω: p form, μ: q-form

Categorified Electromagnetism

ρ is a time dependent 1-form on space, "$\vec{\rho}$"

j is a time-dependent 2-form on space "$\vec{j}$"

$J = -\rho\, dt + j$ 2-form on spacetime

$$*d*F = J$$

(each side is a 2-form)

We get 2 eqns writing

$$F = E\,dt + B$$

3-form, $p=3$; E: 2-form; B: 3-form

$$= dt\,E + B$$

no minus since E is a 2 form

Recall – $dw = d_s w + dt \frac{dw}{dt}$

$$d(uw) = du\,w + (-1)^p u\,dw$$

EM

$$*F = -*_s(-E) + dt *_s B$$

$$= *_s E + dt *_s B$$

$$\underbrace{d(*F)}_{\text{3-form}} = d_s *_s E + dt \frac{d}{dt}(*_s E)$$

$$\frac{d}{dt}(*_s E) = *_s \frac{dE}{dt}$$

$$+ d_s(dt *_s B)$$

$$d_s(dt *_s B) = -dt\, d_s *_s B$$

$$= d_s *_s E + dt *_s \frac{dE}{dt} - dt\, d_s *_s B$$

$$*d*F = -\underbrace{*_s\,(*_s}_{id.} \frac{\partial E}{dt}) + *_s(d_s *_s B)$$

$$+ (-1)^3 dt *_s (d_s *_s E)$$

$$= -\frac{dE}{dt} + *_s d_s *_s B - dt \underbrace{*_s d_s *_s E}_{\text{zero form}}$$

Cat EM

$$*F = -*_s E - dt *_s B$$

$$\underbrace{d(*F)}_{\text{2-form}} = -d_s *_s E$$

$$- dt *_s \frac{dE}{dt} + dt\, d_s *_s B$$

$$*d*F = *_s *_s \frac{dE}{dt}$$

$$- *_s d_s *_s B$$

$$+ (-1)^2 dt *_s (-d_s *_s E)$$

$$= \frac{dE}{dt} - *_s d_s *_s B$$

$$- dt *_s d_s *_s E$$

$\Omega^0(\mathbb{R}^3)$

$\Omega^1(\mathbb{R}^3)$ — E

$\Omega^2(\mathbb{R}^3)$ — B

$\Omega^3(\mathbb{R}^3)$

EM

a 0-form, so we can switch ρ & dt

$$*d*F = J = -dt\rho + j$$

iff $*_s d_s *_s E = \rho$ Gauss's Law
Div. of elect. field is ρ

&

$$-\frac{\partial E}{\partial t} + *_s d_s *_s B = j$$

OR

$$\boxed{\begin{aligned} \vec{\nabla}\cdot\vec{E} &= \rho \text{ and} \\ \vec{\nabla}\times\vec{B} &= \vec{j} + \frac{\partial \vec{E}}{\partial t} \end{aligned}}$$

CAT EM

$$*d*F = J = -\rho dt + j$$
$$= dt\rho + j$$

since ρ is a 1-form and switching them introduces a minus sign

iff

$$*_s d_s *_s E = -\rho$$

and

$$\frac{\partial E}{\partial t} - *_s d_s *_s B = j$$

OR

(now E is a 2-form, b a 3-form)

$$\boxed{\begin{aligned} \vec{\nabla}\times\vec{E} &= -\vec{\rho} \text{ and} \\ -\vec{\nabla}b &= \vec{j} - \frac{\partial \vec{E}}{\partial t} \end{aligned}}$$

Maxwell's Eqns

	Ordinary	Categorified
Trivial	$\vec{\nabla}\cdot\vec{B} = 0$	$0 = 0$
	$\vec{\nabla}\times\vec{E} = -\dfrac{\partial \vec{B}}{\partial t}$	$\vec{\nabla}\cdot\vec{E} = \dfrac{\partial b}{\partial t}$
Non-Trivial	$\vec{\nabla}\cdot\vec{E} = \rho$	$\vec{\nabla}\times\vec{E} = -\vec{\rho}$
	$\vec{\nabla}\times\vec{B} = \dfrac{\partial \vec{E}}{\partial t} + \vec{j}$	$\vec{\nabla} b = \dfrac{\partial \vec{E}}{\partial t} - \vec{j}$

Without matter, ($\rho = 0$, $J = 0$) ordinary EM becomes symmetrical.

$$\left.\begin{array}{l} \vec{E} \longrightarrow \vec{B} \\ \vec{B} \longrightarrow -\vec{E} \end{array}\right] \text{duality}$$

Secretly: $F \longrightarrow *F$

$$\left.\begin{array}{l} dF = 0 \\ d*F = 0 \end{array}\right\} \text{vacuum Maxwell eqns.}$$

F is a 2 form, and
$*F$ is a 2-form (since we're in $\mathbb{R}^4$)

Ordinary EM has duality symmetry, categorified EM does not, because spacetime is 4-dim'l and F is a 2-form in ordinary EM and $4 = 2 + 2$.

In 6-dimensions, but we would have symmetry in cat. EM.

In 6d spacetime categorified EM has duality symmetry: $6 = 3 + 3$.