2/26/02

Maxwell's Eqns

$$dF = 0$$
$$*d*F = J \qquad \text{2-form } F \quad \text{electromagnetic field}$$

First eqn will hold automatically if $F = dA$
· This looks like a trick! Let's understand what A is.

Quest: Is A "fundamental" or is F?

Note: If spacetime is $\mathbb{R}^4$, it doesn't matter which we use (A or F) because any 2-form F w/ $dF = 0$ is of the form dA for some 1-form A.
Now – is A unique?
Well – while A is not unique, any 2 1-forms A, A' with
$$dA = dA'$$
are related by
$$A' - A = df$$
for some 0-form f.

(Easy part: If $A' - A = df$, then
$$dA' - dA = ddf = 0.)$$

Hard part: If $dA = dA'$ then
$$A' - A = df \text{ for some } f.$$
This part depends on the topology of our spacetime.

ie) This fails for spacetimes of other topologies.

Moreover – on same manifolds, $dF = 0 \not\Rightarrow F = dA$

True for $\mathbb{R}^n$: if $dB = 0$ Then $B = d\square$ for some $\square$.

<u>De Rham cohomology</u>: If M is a smooth manifold we get:

$$\Omega^0(M) \xrightarrow{d} \Omega^1(M) \xrightarrow{d} \Omega^2(M) \xrightarrow{d}$$

with $d^2 = 0$, which implies

$$* \quad Z^p(M) = \{\omega \in \Omega^p(M) \mid d\omega = 0\}$$

"cocycles" cycle – boundary is zero.

d is called a '<u>coboundary</u>' map.
d boundary map sends from p-space to p-1 space.

$$* \quad B^p(M) = \{\omega \in \Omega^p(M) \mid \omega = d\alpha \text{ for some } \alpha \in \Omega^{p-1}(M)\}$$

"coboundaries"
(<u>are</u> d of something)

$$d^2 = 0 \Rightarrow B^p(M) \subseteq Z^p(M).$$

but sometimes this is a strict inclusion $\subset$.

let $H^p(M) = \dfrac{Z^p(M)}{B^p(M)}$

"de Rham cohomology"

This tells us something about the holes in space.

If $M = \mathbb{R}^n$, then $H^p(M) = 0 \quad \forall\, p > 0$.

Note: $df = 0 \Rightarrow f$ is constant

But in general $H^p(M)$ is an interesting topological invariant of M — finite dim'l if M compact.

So — for $\mathbb{R}^4$ it "doesn't matter" whether we use A or F, but A is nice if we're interested in...

sidenote: (A = 1-form, integrate along world line of particle — action)

$\boxed{\text{Action}}$

Suppose we have a particle moving along a path in space:

$$\underset{\text{time}}{q : [0, T]} \longrightarrow \underset{\text{space}}{\mathbb{R}^3}$$

We want to know how our particle moves.
We phrase the question as:

Given $q(0)$, $q(T)$, what is the particle's path?
(How does it get from time 0 to time T).

It minimizes some quantity called the <u>action</u>, S, which is of the form

$$S = \int_0^T \mathcal{L}(q(t), \dot{q}(t))\, dt$$

where $\mathcal{L}$ is some function

$$\mathcal{L} : \underset{q}{\mathbb{R}^3} \times \underset{\dot{q}}{\mathbb{R}^3} \longrightarrow \mathbb{R}$$

(or if you replace $\mathbb{R}^3$ by a manifold M,
$\mathcal{L} : TM \longrightarrow \mathbb{R}$.)

tangent bundle of M

Actually the particle will not minimize S, but will merely achieve a critical point:

light - knows how to find crit. pts, not abs max/min

$$\delta S = 0$$

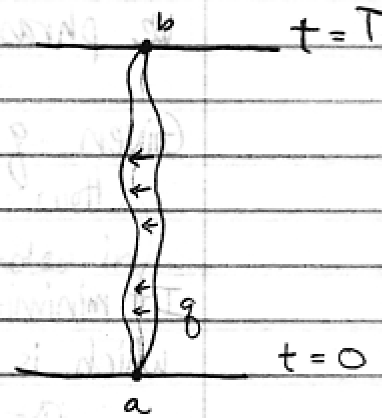
(i.e. $dS = 0$ where S is a function $S : P \longrightarrow \mathbb{R}$

$$P = \{ q : [0, T] \longrightarrow \mathbb{R}^3 \mid q(0) = a,\ q(T) = b \}$$

a path, not particle

where

$$\lim_{\varepsilon \to 0} \frac{S(q+\varepsilon \delta q) - S(q)}{\varepsilon} = \delta S$$



$\delta_q : [0,T] \longrightarrow \mathbb{R}^3$

$\delta q(0) = \delta q(t) = 0$

So — we want $\delta S = 0$
for all δq st $\delta q(0) = \delta q(T) = 0$.

Which paths make $\delta S = 0$ in this sense?

$$0 = \delta S = \delta \int_0^T \mathcal{L}(q(t), \dot{q}(t))\, dt$$

This can change in 2 ways: $q(t)$ or $\dot{q}(t)$

$$= \int_0 \delta \mathcal{L}(q(t), \dot{q}(t))\, dt$$

Einstein sum notation in denominator, upstairs is downstairs for indices

$$= \int_0^T \frac{\partial \mathcal{L}}{\partial q^i} \delta q^i(t) + \frac{\partial \mathcal{L}}{\partial \dot{q}^i} \delta \dot{q}^i(t)\, dt$$

use integration by parts

so write $\delta \dot{q}^i(t) = \delta \frac{dq^i}{dt}(t) = \frac{d}{dt}\delta q^i(t)$

$$= \int_0^T \left(\frac{\partial \mathcal{L}}{\partial q^i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i}\right)\delta q^i \, dt$$

This can only vanish $\forall\ \delta q$ if

$$\underbrace{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{\partial \mathcal{L}}{\partial q^i}}_{\text{Euler-Lagrange eqn.}}$$

Euler-Lagrange eqn.

$$\frac{d}{dt}\underbrace{\frac{\partial \mathcal{L}}{\partial \dot{q}^i}}_{i^{\text{th}}\text{ component of momentum}} = \frac{\partial \mathcal{L}}{\partial q^i}$$

i^{th} component of <u>force</u>

i^{th} component of <u>momentum</u>

<u>Example:</u>

1) <u>Newtonian particle</u> in a <u>potential</u>:

$\mathcal{L}(q(t), \dot{q}(t))$ = Kinetic energy − potential energy

note− action is what is happening
so− Kinetic energy− what's happening
potential energy− what's <u>not</u> happening / <u>could</u> happen

So, for example, a rock on a cliff has lots of potential energy, so negative action.

$\mathcal{L}(q(t), \dot{q}(t))$ = Kinetic energy - potential energy

$$= \frac{m}{2} \underset{\dot{q}^i \dot{q}_i}{\underset{\parallel}{\dot{q}^2}} - V(q(t))$$

$m>0$ mass $\qquad V: \mathbb{R}^3 \longrightarrow \mathbb{R}$ "potential"

$$\text{momentum} = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = m\dot{q}_i$$

$$\frac{d}{dt} \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = m\ddot{q}_i = \text{mass} \cdot \text{acceleration}$$

$$\frac{\partial \mathcal{L}}{\partial q^i} = -\frac{\partial V}{\partial q^i} = \text{force}$$

So we get $F = ma$.

2) Special relativistic particle in electromagnetic field.

Now - we won't view our particle's path as space & time parts.

Now - our particle's path will be

$$q : [0, T] \longrightarrow \mathbb{R}^4$$

"parameter space" $\qquad$ spacetime

Here we're using Minkowski metric: $-dt^2+dx^2+dy^2+dz^2$
Want to minimize distance w/ this metric

and action is:

$$S = -m \cdot \text{length of path} + e\int_q A$$

mass

charge

our path

or "total proper time" of path

$$S = -m\int_0^T \sqrt{-\frac{dq^i}{ds}\cdot\frac{dq_i}{ds}}\, ds$$

particle like this to be big

inner product

$$+\, e\int_0^T A_i(q(s))\frac{dq^i}{ds}\, ds$$

$$\dot{q}^i = \frac{dq^i}{ds}$$

$i = 0,1,2,3$ so, $\mathcal{L}(q,\dot{q}) = -m\sqrt{-\dot{q}^i\dot{q}_i} + e\,A_i(q)\,\dot{q}^i$

$$\text{momentum} = \frac{\partial\mathcal{L}}{\partial\dot{q}^i} = m\cdot\frac{\dot{q}_i}{\sqrt{-\dot{q}^i\dot{q}_i}} + e\,A_i(q)$$