2/28/02

Categorified Gauge Theory

1) Classified strict skeletal 2-groups
2) Classified weak skeletal 2-groups
(has associator – not hold on nose)

To do: 3) Classify weak 2-groups
(drop skeletalness)

Note – every group is a 2-group
(2-morphisms are identities)

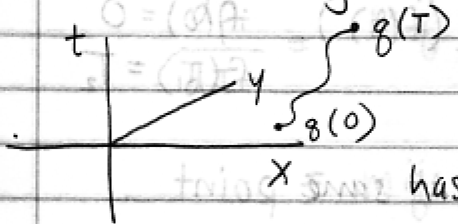
1) The 2-group $G = U(1)$, $A = 1$
$\Rightarrow$ electromagnetism
2) The 2-group $G = 1$, $A = U(1)$ $\Rightarrow$ categorified EM

Should do: 3) The 2-group G = arbitrary compact Lie gp $A = \underline{1}$
$\Rightarrow$ Yang-Mills eqns

4) Strict skeletal Lie 2-group e.g.:
G = compact Lie grp
A = compact abelian Lie grp.
a) Action of G on A is trivial ✓
b) Action of G on A nontrivial

5) General case (w/ associator, etc)

Particle tracing out a path in space time



$q : [0, T] \longrightarrow \mathbb{R}^4$
$\quad\quad\;\; \swarrow$
$\quad\quad\;\; s$

has action

$S(q) = S = m \cdot (\text{length of } q) + e \int A$ ← integral over the path

or

$$S = -m \int_0^T \sqrt{-\dot{q}^i \dot{q}_i}\, ds + e \int_0^T A_i(q(s))\, \dot{q}^i(s)\, ds$$

$-dt^2 + dx^2 + dy^2 + dz^2 \qquad \dot{q} = \frac{dq}{dt}$

Work out E-L eqns:

$$\underbrace{\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}^i}}_{\text{momentum} = p^i} = \underbrace{\frac{\partial \mathcal{L}}{\partial q^i}}_{\text{force}}$$

where $\mathcal{L} = -m\sqrt{-\dot{q}^i \dot{q}_i} + eA_i \dot{q}^i$

Note – the action is independent of the parametrization of the path

S doesn't change when we reparametrize the path!

$S(q_1) = S(q_2)$ if q_1, q_2 paths q_i

$q_1 : [0, T_1] \longrightarrow \mathbb{R}^4$ and $q_2 : [0, T_2] \longrightarrow \mathbb{R}^4$ ↓

are related by $q_1(s) = q_2(f(s))$ $\quad f(0)=0$

$f(T_1) = T_2$

proper time— time measured by a watch moving along the path

We can deal with this by: at same point deciding to parametrize our curve by arclength (proper time — time as measured by a watch moving along the path)

i.e.

st $\sqrt{-\dot{q}_i \dot{q}^i} = 1$

This "nice" choice of s kills off reparametrizations and is called "τ": proper time

$$\text{"length of } q\text{"} = \int_0^T d\tau = T$$

$$\mathcal{L}(q^i, \dot{q}^i) = \mathcal{L} = -m\sqrt{-\dot{q}_i \dot{q}^i} + \underbrace{e A_j \dot{q}^j}_{= e A_j(q)\dot{q}^j}$$

$$p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}^i} = \frac{m\dot{q}_i}{\sqrt{-\dot{q}_i \dot{q}^i}} + eA_i$$

To work out $\frac{d}{ds}\frac{\partial \mathcal{L}}{\partial \dot{q}^i}$, now let's use proper time!

* All these vectors have 4 components *

Then

$$p_i = m\dot{q}_i + eA_i$$

$$\frac{dp_i}{d\tau} = m\ddot{q}_i + e\frac{dA_i}{d\tau}$$

$$= m\ddot{q}_i + e\frac{\partial A_i}{\partial x^j}\dot{q}^j$$

On the other side of E-L eqns get

the 'q' is lurking in the A_i

$$\frac{\partial \mathcal{L}}{\partial q_i} = e\frac{\partial A_j}{\partial x^i}\dot{q}^j$$

So our particle's path satisfies:

$$m\ddot{q}_i + e\frac{\partial A_i}{\partial x^j}\dot{q}^j = e\frac{\partial A_j}{\partial x^i}\dot{q}^j$$

$$m\ddot{q}_i = e\dot{q}^j\underbrace{\left(\frac{\partial A_j}{\partial x^i} - \frac{\partial A_i}{\partial x^j}\right)}_{}$$

this is "curl" or $dA = F$

$$= e\dot{q}^j F_{ij} \quad \text{(Lorenz/Lorentz force law)}$$

since $F = dA$,

$$A = A_i\, dx^i$$

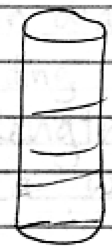
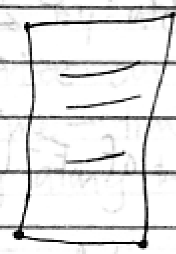
$$dA = \frac{\partial A_i}{\partial x^j}\, dx^j dx^i$$

$$= \frac{1}{2}\left(\frac{\partial A_i}{\partial x^j} - \frac{\partial A_j}{\partial x^i}\right) dx^j dx^i$$

$$= \frac{1}{2} f_{ij}\, dx^i dx^j$$

<u>Moral</u>: The eqns of motion only involve F
but action involves A.

<u>Categorified version</u>:

closed string moving along  or 

replaces the curve $q: [0,T] \longrightarrow \mathbb{R}^4$

Might as well consider $q: \Sigma \longrightarrow \mathbb{R}^4$ where Σ is any compact oriented 2-manifold w/ boundary

The action is:

$$S = -m \cdot \text{Area of } q(\Sigma) + e \int_{q(\Sigma)} A$$

where now A is a 2-form
the Neveu-Schwarz field.
This is the action of a bosonic string theory.

diffeomorphisms S has lots of reparametrization invariance...
Diff (Σ) acts as symmetries.

Green, Schwarz & Witten's book: <u>Superstring Theory</u>
"the classical bosonic string"

Hooke's Law: $F = kx$, $E = \frac{kx^2}{2}$

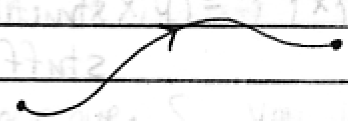
rubber band: has $\begin{cases} \text{potential energy} = m \cdot \text{length of it} \\ \text{kinetic energy} \end{cases}$

we can write Lagrangian as difference of these 2 energies.

Quantization:

We've seen the role of A in classical EM is to compute $\int A$ - integral over curve.

Or - an $\mathbb{R}$-connection



integral over this curve is a real #.

In quantum mechanics, it's different: instead of S, what matters is

$e^{iS/\hbar}$ where $\hbar$ = Planck's constant

Now - we get a $U(1)$ connection: to any path γ we get an element $e^{iS(\gamma)/\hbar} \in U(1)$