2/5/02

A strict skeletal 2-group C amounts to:

- a group G
- an abelian group A
- an action ρ of G on A

weaken associative law – gives us associator

A weak skeletal 2-group (w/ associator instead of strict associativity, but w/ left & right unit laws strict, and inverses strict ie) $gg^{-1} = id$) amounts to:

having the same stuff as before, but now also with

$$\alpha : G^3 \longrightarrow A \qquad \begin{array}{c} (fg)h \\ \Big\downarrow 2 \;\; \alpha_{f,g,h} \\ f(gh) \end{array}$$

(called 3-cocycle cond.) satisfying the pentagon eqn. (and more)

So – we'll define a Lie 2-group to be one of the above (either weak or strict) where:

G is a Lie group
A is an abelian Lie group
ρ is smooth
α is smooth

Lie grp has a Lie algebra — tangent space @ identity, so we should have a similar notion for our Lie-2 group.

So, a Lie 2-algebra will (probably) amount to:

- a Lie algebra $\mathfrak{g}$
- a vector space $\mathfrak{a}$ (an abelian Lie alg)
- a representation $d\rho$ of $\mathfrak{g}$ on $\mathfrak{a}$.
- a linear map $d\alpha : \mathfrak{g}^3 \longrightarrow \mathfrak{a}$ satisfying some equation coming from differentiating the pentagon eqn.

a Lie alg. 3-cocycle

Gauge Theory

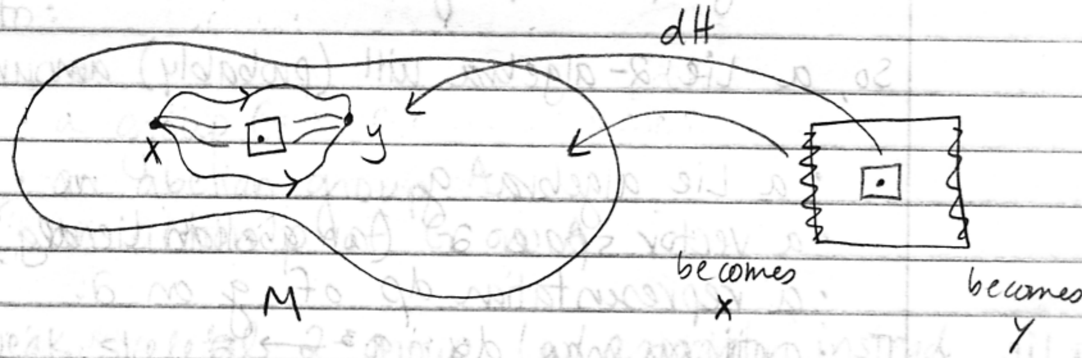
If G is a group and M is a smooth manifold (spacetime), a generalized connection is a functor:

$$\Pi_1^{(\text{thin})}(M) \longrightarrow G$$

"thin fundamental groupoid"

a category w/ one object and morphisms = grp. elts (composition of morphisms is mult. in group)

<u>Recall</u>: fundamental groupoid: make a category
objects = pts
morphisms = paths w/ equiv. relation

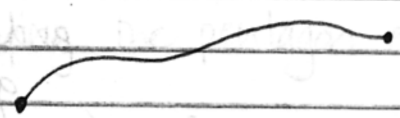


So —

$\Pi_1^{(\text{thin})}(M)$ has $\{\text{objects}\} = M$
and morphisms are "thin homotopy classes" of paths in M. A homotopy between paths

$$H : [0,1]^2 \longrightarrow M$$

is <u>thin</u> if it's smooth and its differential (linear map between tang spaces) dH has rank < 2 everywhere.

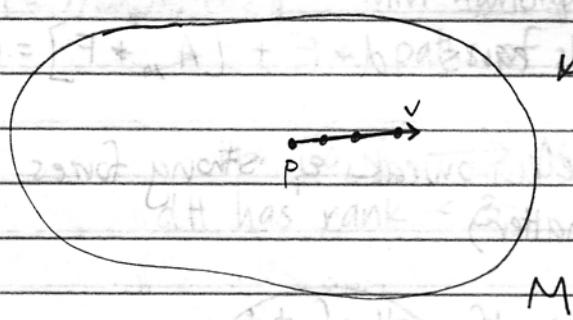


A <u>thin</u> homotopy of this path — slide path along itself (just re-parametrize curve)

<u>Note</u> — we can turn a corner ┐ smoothly — use functs whose deriv = 0 (slow down) stop at corner, then start up smoothly again.

$\Pi_1^{thin}(M)$ is something like a smooth manifold and if G is a Lie group and F is smooth, we call F a <u>smooth</u> <u>connection</u> — equivalent to the usual definition.

If F is smooth, we can differentiate it:



form a 1-parameter family of paths starting w/ the constant path at p and stretching out in v direction as time passes

F assigns to each path a group element

$$F: \Pi_1^{thin}(M) \longrightarrow G.$$

Differentiating F gives us a tang. vector - ie) element of the Lie algebra.

· a tangent vector v at $p \in M$ determines a Lie alg. elt in $\mathfrak{g}$.

we get a map

$$T_pM \longrightarrow \mathfrak{g} \quad \forall p$$

and thus a $\mathfrak{g}$-valued 1-form A - what physicists call the connection

From A we get:

- curvature $F = dA + \frac{1}{2}[A,A]$
- Bianchi identities $dF + [A,F] = 0$
  (Maxwell's eqns are 2 of these)
- Yang-Mills eqns $d{*}F + [A, {*}F] = 0$.

Hodge-$*$ operator

& describes electroweak & strong forces
(w/o matter)

Now - Categorify all of this

Fix a 2-group C and a manifold M.
A "generalized 2-connection" is a 2-functor

$$F : \underbrace{\Pi_2^{\text{thin}}(M)}_{\substack{\text{a 2-category} \\ \text{(weak or strong)}}} \longrightarrow \underbrace{C}_{\text{2-category}}$$

F a 2-functor sends obj $\to$ obj
morph $\to$ morph
2-morph $\to$ 2-morph
sends source/target $\to$ F(source/target)
everything holds on the nose

where $\Pi_2^{thin}(M)$ has:

$\{\text{objects}\}$ = pts in M

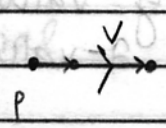
$\{\text{morphisms}\}$ = paths in M (not equiv. classes)

$\{\text{2-morphisms}\}$ = thin homotopy classes of paths of paths

$$H : [0,1]^3 \longrightarrow M$$
dH has rank < 3.

$$\begin{tikzcd} \bullet \arrow[r, bend left] \arrow[r, bend right] \arrow[r, phantom, "\Downarrow"] & \bullet \end{tikzcd}$$

If F is smooth and C is a Lie 2-group, we can differentiate it and get:

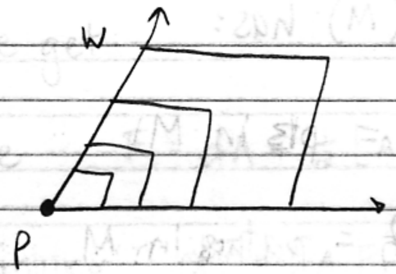


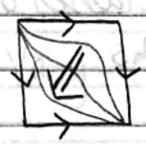
Similar 1-parameter family of paths

$\downarrow$ gives

a $\mathfrak{g}$-valued 1-form A

(we can also see what F does to a 1-parameter family of 2-morphisms)





1-parameter family of paths of paths

Differentiate F, apply to this 1-parameter family of 2-morphisms gives an $\mathfrak{h}$-valued 2-form B.

(in string theory goes by name of Neveu-Schwarz field)

So — a "smooth 2-connection" boils down to a pair $\omega = (A, B)$: $\mathfrak{g}$-valued 1-form and $\mathfrak{h}$-valued 2-form

What if f is a weak 2-functor? Is there more?

Want to understand:

- Curvature of ω: (F, G) w/

$$F = dA + \tfrac{1}{2}[A, A]$$
$$G = dB + \rho(A) \wedge B$$

- Bianchi identities
- Yang-Mills eqns.

Simple examples:

- $G = 1$, $\mathfrak{g} = \{0\}$ $A = U(1)$ $\mathfrak{a} = \mathfrak{u}(1) \cong \mathbb{R}$

Here - all we get is a 2-form B - the Neveu-Schwarz field in string theory?

$$\text{Curvature} = G = dB \qquad \text{(a 3-form)}$$

$$\text{Bianchi identities:} \quad dG = ddB = 0$$

$$\text{Yang-Mills eqns:} \quad d * G = 0$$

"2-form electromagnetism"

EM	Categorified EM
1 form A - vector potential	2-form B
2 form - $F = dA$ electric & magnetic fields	3-form $G = dB$
$dF = 0$, $d{*}F = 0$ } Maxwell's eqns.	$dG = 0$, $d{*}G = 0$

point particle

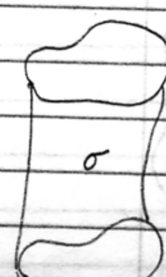
action

$$S = \int_\gamma A \in \mathbb{R}$$

γ

$$e^{is} \in U(1)$$

string



$$S = \int_\sigma B$$

$$e^{is} \in U(1)$$

two-forms integrated over surfaces