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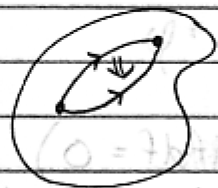


M

manifold

G

groupoid

We get a 2-groupoid from M where objects – pts
morphisms – paths
2-morphisms – paths of paths

$\Pi_2^{\text{thin}}(M) \xrightarrow[\text{connection}]{} C$ 2-group

- w/ morphisms forming group G
- w/ 2-morphisms $\alpha : 1 \Rightarrow 1$ forming abelian grp A.

A = trivial, G = U(1) } electromagnetism

G = trivial, A = U(1) } categorified EM

How does a 1-form A (electromagnetism)
or 2-form A (cat. EM)
define a connection of this sort?

∂x_1 $\quad$ dx_n

Integration of n-forms on $\mathbb{R}^n$:

On $\mathbb{R}^n$, any n-form ω looks like:

$$\omega = f\,dx_1 \cdots dx_n$$

(recall $df\,df = 0$) since having repeated dx_i will get zero.

so we define:

$$\int_{\mathbb{R}^n} \omega = \int_{\mathbb{R}^n} f\, dx_1\, dx_2 \ldots dx_n$$

Now suppose $\varphi : M \longrightarrow N$ is a smooth function between smooth manifolds. Then we can pull back smooth functions:

$$\varphi^* : \underbrace{C^\infty(N)}_{\text{smooth functs on } N} \longrightarrow C^\infty(M)$$

defined by

$$\varphi^*(f)(x) = f(\varphi(x)) \qquad x \in M$$

$$\begin{tikzcd} & & \mathbb{R} \\ M \arrow[urr, "\varphi^* f = f \circ \varphi"] \arrow[rr, "\varphi"'] & & N \arrow[u, "f"'] \end{tikzcd}$$

This gives us a unique alg. homo:

$$\varphi^* : \Omega(N) \longrightarrow \Omega(M)$$

diff forms on N

such that:

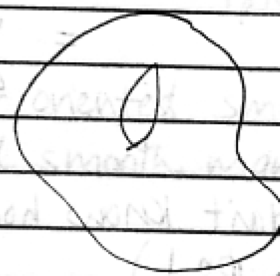
algebra generated by df's

$$\varphi^* : f \longmapsto \varphi^*(f)$$
$$df \longmapsto d\varphi^*(f)$$

this defn. tells us what happens to anything in $\Omega(N)$ since we know what it does to the generators df.

Integration of n-forms over an n-dim'l manifold M:

We know how to integrate on $\mathbb{R}^n$, so, we use the fact that M is a manifold so has parts homeo. to $\mathbb{R}^n$.
M - covered by U_i's st
$\varphi_i : \mathbb{R}^n \longrightarrow U_i$ homeos.



M

we want to take diff. forms on each U_i & pull them back to $\mathbb{R}^n$ & integrate there!

To reduce integration on M to integration on $\mathbb{R}^n$ find smooth functions $\psi_i : M \longrightarrow \mathbb{R}$ w/ supp $\psi_i \subseteq U_i$ st

$$\sum_i \psi_i = 1 \quad \text{(partition of unity)}$$

and write $\omega \in \Omega^n(M)$ as

$$\omega = \omega \sum \psi_i$$

$$= \sum \underbrace{\omega \psi_i}_{\omega_i}$$

where ω_i is zero outside of U_i.
(so — we just use ω_i on U_i).

Then we define:

$$\int_M \omega = \sum_i \int_M \omega_i$$

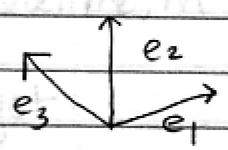
$$= \sum_i \int_{\mathbb{R}^n} \varphi_i^* \omega_i$$

(we don't know how to integrate on M, but we do on $\mathbb{R}^n$!)

<u>Thm</u>: $\int_M \omega$ doesn't depend on our parametrization U_i, φ_i, ψ_i

as long as we fix an orientation on M and $\mathbb{R}^n$ and demand that all φ_i are orientation-preserving.

In 3d - orientation is right/left hand rule.

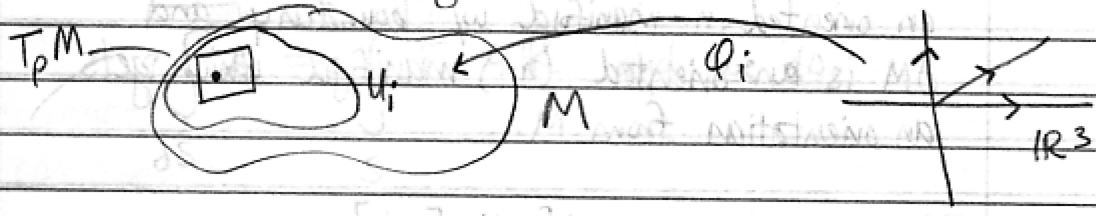


e_1, e_2, e_3 is "right-handed" basis of vectors in $\mathbb{R}^3$

we have also "left-handed" basis.

These are orientations on $\mathbb{R}^3$.

On our manifold, at each pt we have a 3-dim'l v. space called tangent space.

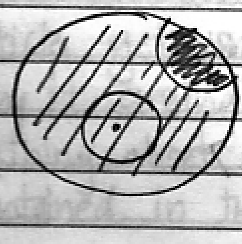


$d\varphi_i : \mathbb{R}^3 \longrightarrow T_p U_i$

pick an orientation for $\mathbb{R}^3$, then demand that orientation of T_pM agrees.

All this works not just for oriented smooth manifolds but also oriented smooth manifolds w/ boundary.

ex)



manifold w/ boundary - small patches look like $\mathbb{R}^n$ or $\mathbb{H}^n$.

Stokes' Thm:
$$\int_M d\omega = \int_{\partial M} \omega$$

where ω is an $(n-1)$ form (so d of it is an n-form) and M an n-manifold.

ω is a compactly supported $(n-1)$ form, M is an oriented n-manifold w/ boundary and ∂M is an oriented $(n-1)$ manifold which gets an orientation from M.

Ex) This gives us, if $M = [a,b]$
$$\int_a^b \frac{df}{dx}\, dx = f(b) - f(a)$$

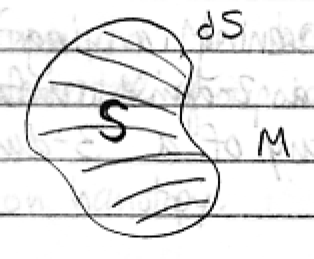
or – in statement of Thm, replace
$$\int_M \omega \text{ by } \langle M, \omega \rangle, \quad d \text{ by } d^*$$

then Stokes' Thm says

(there's an adjoint here)
$$\langle M, d\omega \rangle = \langle d^* M, \omega \rangle$$

Electromagnetism: we have a 1-form A on spacetime, M (e.g. $\mathbb{R}^4$), and $F = dA$, a 2-form.

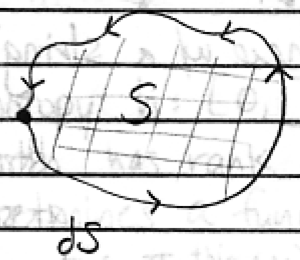
We get:



we expect to get

$$\underbrace{q\int_{\partial S} A}_{\text{action}} = q\int_S F \in \mathbb{R}$$ and we do.

action for carrying a particle around a loop.



In QM — we care about $e^{i \cdot \text{Action}/\hbar} \in U(1)$ which we call a "phase." The state of a particle gets multiplied by this phase when we carry it around the loop ∂S.

Everything about how particles feel EM force is contained in these phases.

In categorified EM, we get this picture:

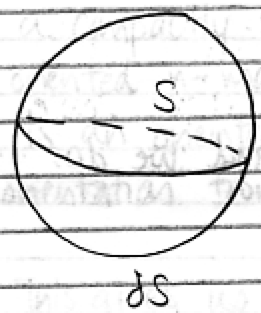
$$\int_{\partial S} A = \int_S F \in \mathbb{R}$$

∂S = 2-dim'l boundary of S.

where A is a 2-form, $F = dA$ is a 3-form, S a 3-dim'l manifold w/ bound.

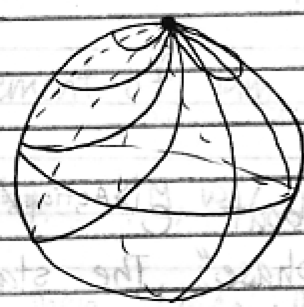
EM – ∂S meant carrying a pt particle around loop.

cat EM – ∂S meaning carrying a string around a 2-dim'l surface that is the boundary of a 3-dim'l thing.



S is a 3-ball in $\mathbb{R}^4$
∂S – 2 sphere

how do we trace out a sphere w/ a string? start w/ it at N. pole



1st starts & ends at same place