3/7/02

EM

Spacetime $M \simeq \mathbb{R} \times S$ ← time, ← space

metric $-dt^2 + g$ ← metric on S

$a = -dt \wedge \phi + A$ U(1) connection 1-form (real-valued)

$\phi(t) \in \Omega^0(S)$, $A(t) \in \Omega^1(S)$ (time-dependent)

$$F = d_M a = dt \wedge (\dot{A} + d\phi) + dA = -dt \wedge E + B$$

$$\begin{cases} E = -d\phi - \dot{A} \\ B = dA \end{cases}$$

exterior deriv on M (spacetime) → $d_M = dt \wedge \frac{\partial}{\partial t} + d$ ← exterior deriv. on space

$$d = dx \wedge \frac{\partial}{\partial x} + \dots + dz \wedge \frac{\partial}{\partial z}$$

Hodge $*$: $*_M F = dt \wedge *B + *E$

$$d_M *_M F = 0$$

$$dt \wedge (*\dot{E} - d*B) + d*E = 0$$

$$*_M d_M *_M F = J$$

are zero

use the fact that in 3d, $*^2 = 1$.

$$*_M d_M *_M F = -dt \wedge *d*E + *d*B - *^2 \dot{E}$$

$*d*E = 0$ ie. $\operatorname{div} \vec{E} = 0$

$*d*B = \dot{E}$ ie. $\operatorname{curl} \vec{B} = \dot{E}$

} Maxwell's eqns in a vacuum.

categorified EM

$a = dt \wedge \phi + A$ 2-form

$F = d_M a = dt \wedge E + B$

$*_M d_M *_M F = 0 \Rightarrow \begin{cases} *d*E = 0 & \text{curl } \vec{E} = 0 \\ *d*B = \dot{E} & \text{grad } \vec{B} = \dot{\vec{E}} \end{cases}$

$\begin{cases} E \text{ a 2-form} \\ B \text{ a 3-form} \end{cases}$

"p-form" EM

$a = dt \wedge (-1)^p A_0 + A$ is a p-form
$F = d_M a$
$*_M d_M *_M F = 0$

Hodge Theory

$\Omega^p(S)$ – space of p-forms on S
which has an inner product on it:

$$\langle A, B \rangle := \int_S g(A,B)\,\text{vol} \quad \text{where } A, B \in \Omega^p(S)$$

$g(X, \cdot) =$ some 1-form α_X. $\quad g: V(S) \times V(S) \to \Omega^0(S)$
↑ vect. fields

Define $g(\alpha_X, \alpha_Y) = g(X, Y)$

$$g : V(S) \longrightarrow \Omega^1(S)$$
$$X \longmapsto g(X,\cdot)$$

$$g^{-1} : \Omega^1(S) \longrightarrow V(S)$$

$$g^{-1} : \Omega^1(S) \times \Omega^1(S) \longrightarrow \Omega^0(S)$$

Define $g(\alpha_1 \wedge \dots \wedge \alpha_p ; \beta_1 \wedge \dots \wedge \beta_p) = \det[g(\alpha_i, \beta_j)]$ $i,j = 1,\dots,p$

we call the inner product of these 2 p-forms the det of the matrix

"timelike" Then $g(dt \wedge dx, dt \wedge dx) =$

$$\det \begin{pmatrix} g(dt,dt) & g(dt,dx) \\ g(dx,dt) & g(dx,dx) \end{pmatrix} = \det \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} = -1$$

like area $g(dy \wedge dz, dy \wedge dz) = \det \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = 1$

we have a metric

$$g : \Omega^p(S) \times \Omega^p(S) \longrightarrow \Omega^0(S)$$

$$g : \Omega^p(S) \longrightarrow (\Omega^p)^*$$

$*: \Omega^p(S) \longrightarrow \Omega^{n-p}(S)$ Hodge star

$g(A,B)\,\mathrm{vol} \in \Omega^n(S)$

$$g(\ \ , B)\,\mathrm{vol} : \Omega^p(S) \longrightarrow \Omega^n$$

$A, B \in \Omega^p(S)$

$\boxed{A \wedge *B = g(A,B)\,\mathrm{vol}}$ this defines the action of $*$ on p-forms

$\mathrm{vol} = dx \wedge dy \wedge dz$ in space

$dx \wedge \boxed{(*dx)} = g(dx, dx)\,\mathrm{vol} = dx \wedge \boxed{dy \wedge dz}$

so $*dx = dy \wedge dz$

Properties of $*$:

$*^2 = (-1)^{p(n-p)} : \Omega^p(S) \longrightarrow \Omega^{n-p}(S)$ in Riemannian geo.

$*^2 = -(-1)^{p(n-p)}$ in Lorentzian geo.

$*dt = dx \wedge dy \wedge dz$

$g(A,B) = g(*A, *B)$ in Riemannian signature

$= -g(*A, *B)$ in Lorentzian signature $(- + + +)$

So we have

$$\Omega^{p-1}(S) \underset{d^+}{\overset{d}{\rightleftarrows}} \Omega^p(S) \underset{d^+}{\overset{d}{\rightleftarrows}} \Omega^{p+1}(S)$$

Hilbert spaces (have an inner product)

d^+ would be defined by $\langle dA, B \rangle = \langle A, d^+ B \rangle$

d^+ is related to $*d*$.

$$\begin{tikzcd}
\Omega^{p-1} \arrow[r, "d"] \arrow[d, leftrightarrow, "*"'] & \Omega^{p} \arrow[r, "d"] \arrow[d, leftrightarrow, "*"] & \Omega^{p+1} \arrow[l, bend left, "*d*"] \arrow[d, leftrightarrow, "*"] \\
\Omega^{n+1-p} & \Omega^{n-p} \arrow[l, "d"'] & \Omega^{n-1-p} \arrow[l, "d"']
\end{tikzcd}$$

let $\underline{\delta} = d^+ = -(-1)^{(p-1)(n-p)} *d* \text{ in } \Omega^p(S)$

$$\begin{tikzcd}
\Omega^{p-1}(S) \arrow[r, shift left, "d_{p-1}"] & \Omega^{p}(S) \arrow[l, shift left, "\delta_p"] \arrow[r, shift left, "d_p"] & \Omega^{p+1}(S) \arrow[l, shift left, "\delta_{p+1}"]
\end{tikzcd}$$

Kodaira decomposition (decompose Ω^p into direct sum)

d, δ are self-adjoint to each other, so

orthogonal direct sum

$$\Omega^p(S) = \text{Range } d_{p-1} \oplus \ker \delta_p$$

$$\Omega^p(S) = \text{Range } \delta_{p+1} \oplus \ker d_p$$

$$d_p d_{p-1} = 0, \qquad \delta_p \delta_{p+1} = 0$$

Thus, the Kodaira decomposition

$$\Omega^p(S) = \underbrace{\text{Range } \delta \oplus \overbrace{\ker(\underbrace{\delta d + d\delta}_{\text{Hodge Laplacian } -\Delta}) \oplus \text{Range } d}^{\ker d}}_{\ker \delta}$$

Hodge's Thm: $\text{Ker}\,(\delta d + d\delta) \cong H^p(S)$ (p^{th} deRham cohomology class)

Kernel of Laplacian is $H^p(S)$.

If $A \in H^p(S)$ then $dA = 0$ but $A \neq d\phi \ \forall \phi$.

if $A \in \text{Ker}(\delta d + d\delta)$ then $(\delta d + d\delta)A = 0$

so

$$(A, (\delta d + d\delta) A) = 0$$

iff

$$(A, \delta d A) + (A, d\delta A) = 0$$ now integrate by parts

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$d_M *_M F = 0$ interesting half of Maxwell's eqns.

$F = d_M a$ boring half

Action for electromagnetism $S[a] = -\frac{1}{2}\int_M F \wedge *_M F$

where a is any p-form

and $F = d_M a$

$$\delta S[\delta a] = -\int (\delta F) \wedge *_M F = -\int d_M(\delta a) \wedge *_M F$$

we int. by parts $= -\int (\delta a) \wedge d_M *_M F = 0$

$$a = dt \wedge (-1)^p A_0 + A$$

$$d_M a = F = dt \wedge (-1)^p E + B$$

$$S[A_0, A] = \frac{1}{2}\int dt \underbrace{\int_S [g(E,E) - g(B,B)]\, vol}$$

w/ $\frac{1}{2}$ is Lagrangian on space

$$S = \int dA\, L(q, \dot{q})$$

we can write the eqns of motion:

$$\begin{cases} \delta E = 0 & p=1 \quad \text{EM} \\ (-1)^p \dot{E} + \delta B = 0 & p=2 \quad \text{Cat. EM} \end{cases}$$

$$E = (-1)^p \dot{A} - dA_0$$
$$B = dA$$

So we can rewrite these eqns as

Time evolution eqns. for (A_0, A)

$$\begin{cases} \delta(\dot{A} - (-1)^p A_0) = 0 \\ \ddot{A} - (-1)^p d\dot{A}_0 + \delta d A = 0 \end{cases}$$

$\dot{A}_0$ does not appear

$a \longmapsto a - d_M f$ gauge transf.

$A \longmapsto A - df$

$A_0 \longmapsto A_0 - (-1)^p \dot{f}$ f time-independent