k-Colorings as Categorified Coherent States

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A *k*-**coloring** *of a set* S *is a function* $f: S \to k$, *where* k *stands for the* k-*element set. Let* $C(k)$ *be the structure type of 'being a* k-*colored finite set', so that* $C(k)_n$ *is the set of ways of* k-*coloring the* n-*element set.*

1. Compute the cardinality $|C(k)_n|$.

2. Compute the generating function for k-colorings,

$$|C(k)|(z) := \sum_{n \geq 0} \frac{|C(k)|_n}{n!} z^n.$$

Let A *be the annihilation operator on structure types. Thus, to put an* $A\Psi$-*structure on the finite set* S *is to put a* Ψ-*structure on* $S + 1$. *Recall that*

$$|A\Psi| = a|\Psi|$$

where a *is the annihilation operator on formal power series:*

$$(a\psi)(z) = \frac{d}{dz}\psi(z)$$

for any $\psi \in \mathbb{C}[[z]]$.

3. Show that $|C(k)|$ is an eigenvector of the annihilation operator on formal power series, with

$$a|C(k)| = k|C(k)|.$$

4. Determine *all* eigenvectors of the annihilation operator on formal power series. What is special about the eigenvectors coming from k-colorings in part 3?

5. Categorifying part 3, show that $C(k)$ is an eigenvector of the annihilation operator on structure types. In other words, construct an isomorphism of structure types

$$AC(k) \cong kC(k).$$

In the quantum mechanics of a particle on the line, eigenvectors of the annihilation operator are called **coherent states**. *So, you've just seen that the structure type 'being a* k-*colored finite set' is a categorified version of a coherent state!*

But what's the physical meaning *of coherent states? Now we'll show that these are states* ψ *that minimize the product of the uncertainties in position and momentum, attaining equality in the Heisenberg uncertainty principle, which says:*

$$\Delta_\psi q \, \Delta_\psi p \geq \frac{1}{2}$$

in units where $\hbar = 1$. *Coherent states not the only states that do this, but they're the only states that do it while keeping the uncertainty in position* equal *to the uncertainty in momentum:*

$$\Delta_\psi q = \Delta_\psi p$$

Before plunging in, recall that the position and momentum operators are given by

$$q = \frac{a + a^*}{\sqrt{2}}, \qquad p = \frac{a - a^*}{\sqrt{2}i},$$

respectively, where in the Fock representation the annihilation operator a and creation operator a^* are given by

$$(a\psi)(z) = \frac{d}{dz}\psi(z), \qquad (a^*\psi)(z) = z\psi(z)$$

for any $\psi \in \mathbb{C}[[z]]$. Using this we'll work out the inner product in the Fock representation, and use that to normalize the coherent states found in part 4.

6. We choose the inner product on the Fock representation so that

$$\langle 1, 1 \rangle = 1.$$

Note also that

$$z^n = (a^*)^n 1.$$

Using just these together with the fact that a and a^* are adjoint operators and the fact that $[a, a^*] = 1$, show that

$$\langle z^n, z^m \rangle = \delta_{n,m} n!$$

for all $n, m \geq 0$.

7. Take any coherent state ψ from part 4 and write it as

$$\psi(z) = \sum_{n \geq 0} a_n z^n.$$

Use part 6 to compute

$$\langle \psi, \psi \rangle = \|\psi\|^2$$

and use this to find a beautiful simple formula for an arbitrary *normalized* coherent state

$$\frac{\psi}{\|\psi\|}$$

as a function of z.

Recall that in quantum mechanics we use unit vectors to describe states, so we should normalize ψ as in part 7. Henceforth let's switch notation and use ψ to stand for an arbitrary normalized coherent state. I hope this isn't too traumatic.

8. Compute the expected values of position and momentum:

$$\langle \psi, q\psi \rangle, \qquad \langle \psi, p\psi \rangle$$

for your arbitrary normalized coherent state ψ. Show that you can get any values you like if you pick the right ψ.

Hint: since your coherent state is an eigenvector of the annihilation operator, and you know the eigenvalue, it pays to use tricks like this:

$$\langle \psi, q\psi \rangle = \langle \psi, \frac{a + a^*}{\sqrt{2}}\psi \rangle = \frac{1}{\sqrt{2}}\left(\langle \psi, a\psi \rangle + \langle a\psi, \psi \rangle\right)$$

9. Compute the expected values of position and momentum squared:

$$\langle \psi, q^2 \psi \rangle, \qquad \langle \psi, p^2 \psi \rangle$$

for your arbitrary normalized coherent state ψ. (Don't forget the tricks used in part 8.)

10. Compute the variance of position and momentum:

$$
\begin{aligned}
(\Delta_\psi q)^2 &:= \langle \psi, q^2 \psi \rangle - (\langle \psi, q\psi \rangle)^2 \\
(\Delta_\psi p)^2 &:= \langle \psi, p^2 \psi \rangle - (\langle \psi, p\psi \rangle)^2
\end{aligned}
$$

for your arbitrary normalized coherent state ψ.

11. Show that the standard deviations $\Delta_\psi q$ and $\Delta_\psi p$ are equal and their product is $1/2$, the minimum allowed by the Heisenberg uncertainty principle!

12. *Extra credit:* Any observations?