QUANTUM GRAVITY HOMEWORK 4

ERIN PEARSE

1. To compute the number of ways to k-colour an n-element set, observe that there are k possible choices for each element. Since all these choices are independent and repetitions are allowed,
$$|C(k)_n| = k^n.$$

2. Since $|C(k)_n| = |C(k)|_n$, the generating function for k-colourings is
$$|C(k)|(z) = \sum_{n\geq 0} \frac{|C(k)|_n}{n!} z^n = \sum_{n\geq 0} \frac{k^n}{n!} z^n = \sum_{n\geq 0} \frac{(kz)^n}{n!} = e^{kz}$$

3. Using $(a\psi)(z) = \frac{d}{dz}\psi(z)$, we compute
$$(a|C(k)|)(z) = \frac{d}{dz} e^{kz} = ke^{kz} = k|C(k)|(z),$$
thus showing that $a|C(k)| = k|C(k)|$.

4. To determine the eigenvectors of the annihilation operator on formal power series, we consider those
$$f(z) = \sum_{n\geq 0} \frac{a_n}{n!} z^n$$
which have the property that $\frac{d}{dz}f(z) = kf(z)$. By the definition (and uniqueness) of the exponential function, all such functions f will be of the form
$$f(z) = ce^{kz} \quad \text{for any nonzero } c, k \in \mathbb{C}.$$
The eigenvectors which come from k-colourings are precisely those eigenvectors with real eigenvalues.

5. To show that $C(k)$ is an eigenvector of the annihilation operator on structure types, we consider $AC(k)$. Let S be the n-element set. Then putting an $AC(k)$-structure on n is really just putting a $C(k)$-structure on the finite set $n+1$, that is, k-colouring an $(n+1)$-element set. But clearly, k-colouring an $(n+1)$-element set is equivalent to k-colouring n of the elements, and then choosing one of the k colours for the final remaining element. This is the same thing as chopping $n+1$ into two parts, putting the structure of "being a 1-element set which has been assigned one colour (from a set of k colours)" on the first part, and putting a k-colouring on the second part.

The structure type of "being a 1-element set" is Z, but when we allow this element to be coloured with one of k colours, the structure type becomes kZ. Hence, by the product rule for structure types,

$$AC(k) \cong kC(k).$$

6. Now we show that $\langle z^n, z^m \rangle = \delta_{n,m} n!$

$$\begin{aligned}
\langle z^n, z^m \rangle &= \langle (a^*)^n 1, (a^*)^m 1 \rangle \\
&= \begin{cases} \langle a^m (a^*)^n 1, 1 \rangle, & m > n; \\ \langle a^n (a^*)^n 1, 1 \rangle, & m = n; \\ \langle 1, a^n (a^*)^m 1 \rangle, & m < n \end{cases} && a,\ a* \text{ are adjoint} \\
&= \begin{cases} \langle a^{m-n} a^n z^n, 1 \rangle, & m > n; \\ \langle a^n z^n, 1 \rangle, & m = n; \\ \langle 1, a^{n-m} a^m z^m \rangle, & m < n \end{cases}
\end{aligned}$$

Note that in the first case, $m - n \geq 1$ and in the third case, $n - m \geq 1$.

$$\begin{aligned}
&= \begin{cases} \langle a^{m-n} n!, 1 \rangle, & m > n; \\ \langle n!, 1 \rangle, & m = n; \\ \langle 1, a^{n-m} n! \rangle, & m < n \end{cases} && az^n = nz^{n-1} \implies a^n z^n = n! \\
&= \begin{cases} \langle 0, 1 \rangle, & m > n; \\ n! \langle 1, 1 \rangle, & m = n; \\ \langle 1, 0 \rangle, & m < n \end{cases} && ac = \frac{d}{dz} c = 0 \\
&= \begin{cases} 0, & m > n; \\ n!, & m = n; \\ 0, & m < n \end{cases} \\
&= \delta_{n,m} n!
\end{aligned}$$

7. For any nonzero $k, c \in \mathbb{C}$ we have the coherent state

$$\psi(z) = ce^{kz} = c \sum_{n \geq 0} \frac{k^n}{n!} z^n = \sum_{n \geq 0} \frac{ck^n}{n!} z^n.$$

Using this, we compute

$$\begin{aligned}
\langle \psi, \psi \rangle &= \left\langle \sum_{n \geq 0} \frac{ck^n}{n!} z^n, \sum_{n \geq 0} \frac{ck^n}{n!} z^n \right\rangle \\
&= \sum_{n \geq 0} \frac{\overline{c} \overline{k}^n}{n!} \left\langle z^n, \sum_{n \geq 0} \frac{ck^n}{n!} z^n \right\rangle \\
&= \overline{c} \sum_{n \geq 0} \frac{\overline{k}^n}{n!} \sum_{n \geq 0} \frac{ck^n}{n!} \langle z^n, z^n \rangle
\end{aligned}$$

$$
\begin{aligned}
&= \bar{c} \sum_{n\geq 0} \frac{\bar{k}^n}{n!} \sum_{m\geq 0} \frac{ck^m}{m!} \langle z^n, z^m \rangle && \text{changing dummy variable} \\
&= |c|^2 \sum_{m,n\geq 0} \frac{\bar{k}^n}{n!} \frac{k^m}{m!} \langle z^n, z^m \rangle && \text{collecting terms} \\
&= |c|^2 \sum_{n\geq 0} \frac{\bar{k}^n}{n!} \frac{k^n}{n!} n! && \langle z^n, z^n \rangle = \delta_{n,m} n! \\
&= |c|^2 \sum_{n\geq 0} \frac{|k|^{2n}}{n!} \\
&= |c|^2 e^{|k|^2}
\end{aligned}
$$

Thus, $\|\psi(z)\| = \|ce^{kz}\| = |c|e^{|k|^2/2}$. So

$$
\frac{\psi}{\|\psi\|} = \frac{ce^{kz}}{|c|e^{|k|^2/2}} = e^{\arg c} e^{kz - |k|^2/2}
$$

which is just $e^{kz-|k|^2/2}$ up to a phase. Is this really beautiful?

8. We compute the expected value of position as

$$
\begin{aligned}
\langle \psi, q\psi \rangle &= \langle \psi, \frac{a + a^*}{\sqrt{2}} \psi \rangle \\
&= \frac{1}{\sqrt{2}} \left(\langle \psi, a\psi \rangle + \langle a\psi, \psi \rangle \right) \\
&= \frac{1}{\sqrt{2}} \left(\langle \psi, k\psi \rangle + \langle k\psi, \psi \rangle \right) && a\psi = k\psi \\
&= \frac{1}{\sqrt{2}} \left(k\langle \psi, \psi \rangle + \bar{k}\langle \psi, \psi \rangle \right) && \text{sesquilinearity} \\
&= \frac{k + \bar{k}}{\sqrt{2}} \langle \psi, \psi \rangle \\
&= \frac{2\,\mathrm{Re}\,k}{\sqrt{2}} 1 && \psi \text{ is normalized} \\
&= \sqrt{2}\,\mathrm{Re}\,k
\end{aligned}
$$

We compute the expected value of momentum as

$$
\begin{aligned}
\langle \psi, p\psi \rangle &= \langle \psi, \frac{a-a^*}{i\sqrt{2}}\psi \rangle \\
&= \frac{1}{i\sqrt{2}} \left(\langle \psi, a\psi \rangle - \langle a\psi, \psi \rangle \right) \\
&= \frac{1}{i\sqrt{2}} \left(\langle \psi, k\psi \rangle - \langle k\psi, \psi \rangle \right) && a\psi = k\psi \\
&= \frac{1}{i\sqrt{2}} \left(k\langle \psi, \psi \rangle - \overline{k}\langle \psi, \psi \rangle \right) && \text{sesquilinearity} \\
&= \frac{k-\overline{k}}{i\sqrt{2}} \langle \psi, \psi \rangle \\
&= \frac{2i \operatorname{Im} k}{i\sqrt{2}} 1 \\
&= \sqrt{2} \operatorname{Im} k
\end{aligned}
$$

Thus, by choosing k carefully, we can make the expectation of position and momentum be any pair of real numbers. But, we can only categorify the resulting state when k is a natural number.

9. Now for our normalized ψ, we compute the expected value of position squared

$$
\begin{aligned}
\langle \psi, q^2\psi \rangle &= \left\langle \psi, \left(\tfrac{a+a^*}{\sqrt{2}} \right)^2 \psi \right\rangle \\
&= \tfrac{1}{2} \left(\langle \psi, a^2\psi \rangle + \langle \psi, a^* a\psi \rangle + \langle \psi, aa^*\psi \rangle + \langle \psi, (a^*)^2\psi \rangle \right) \\
&= \tfrac{1}{2} \left(\langle \psi, a^2\psi \rangle + \langle a\psi, a\psi \rangle + \langle \psi, (1+a^*a)\psi \rangle + \langle a^2\psi, \psi \rangle \right) \\
&= \tfrac{1}{2} \left(\langle \psi, a^2\psi \rangle + \langle a\psi, a\psi \rangle + \langle \psi, \psi \rangle + \langle a\psi, a\psi \rangle + \langle a^2\psi, \psi \rangle \right) \\
&= \tfrac{1}{2} \left(\langle \psi, k^2\psi \rangle + \langle k\psi, k\psi \rangle + \langle \psi, \psi \rangle + \langle k\psi, k\psi \rangle + \langle k^2\psi, \psi \rangle \right) \\
&= \tfrac{1}{2} \left(k^2\langle \psi, \psi \rangle + |k|^2\langle \psi, \psi \rangle + \langle \psi, \psi \rangle + |k|^2\langle \psi, \psi \rangle + \overline{k^2}\langle \psi, \psi \rangle \right) \\
&= \tfrac{1}{2} \left(k^2 + 2|k|^2 + \overline{k}^2 + 1 \right) \langle \psi, \psi \rangle \\
&= \tfrac{1}{2} \left((k + \overline{k}^2) + 1 \right) 1 \\
&= \tfrac{1}{2} \left((2 \operatorname{Re} k)^2 + 1 \right) \\
&= 2(\operatorname{Re} k)^2 + \tfrac{1}{2}
\end{aligned}
$$

and we compute the expectation of momentum squared

$$\begin{aligned}
\langle\psi, p^2\psi\rangle &= \left\langle\psi, \left(\tfrac{a-a^*}{i\sqrt{2}}\right)^2\psi\right\rangle \\
&= -\tfrac{1}{2}\left(\langle\psi, a^2\psi\rangle - \langle\psi, a^*a\psi\rangle - \langle\psi, aa^*\psi\rangle + \langle\psi, (a^*)^2\psi\rangle\right) \\
&= -\tfrac{1}{2}\left(\langle\psi, a^2\psi\rangle - \langle a\psi, a\psi\rangle - \langle\psi, (1+a^*a)\psi\rangle + \langle a^2\psi, \psi\rangle\right) \\
&= -\tfrac{1}{2}\left(\langle\psi, a^2\psi\rangle - \langle a\psi, a\psi\rangle - \langle\psi, \psi\rangle - \langle a\psi, a\psi\rangle + \langle a^2\psi, \psi\rangle\right) \\
&= -\tfrac{1}{2}\left(\langle\psi, k^2\psi\rangle - \langle k\psi, k\psi\rangle - \langle\psi, \psi\rangle - \langle k\psi, k\psi\rangle + \langle k^2\psi, \psi\rangle\right) \\
&= -\tfrac{1}{2}\left(k^2\langle\psi, \psi\rangle - |k|^2\langle\psi, \psi\rangle - \langle\psi, \psi\rangle - |k|^2\langle\psi, \psi\rangle + \overline{k^2}\langle\psi, \psi\rangle\right) \\
&= -\tfrac{1}{2}\left(k^2 - 2|k|^2 + \overline{k^2} - 1\right)\langle\psi, \psi\rangle \\
&= -\tfrac{1}{2}\left((k-\overline{k})^2 - 1\right)1 \\
&= -\tfrac{1}{2}\left((2i\operatorname{Im} k)^2 - 1\right) \\
&= 2(\operatorname{Im} k)^2 + \tfrac{1}{2}
\end{aligned}$$

10. Now we compute the variance of position and momentum:

$$\begin{aligned}
(\Delta_\psi q)^2 &= \langle\psi, q^2\psi\rangle - (\langle\psi, q\psi\rangle)^2 \\
&= \left(2|k|^2 + \tfrac{1}{2}\right) - \left(\sqrt{2}\operatorname{Re} k\right)^2 \\
&= 2(\operatorname{Re} k)^2 + \tfrac{1}{2} - 2(\operatorname{Re} k)^2 \\
&= \tfrac{1}{2}
\end{aligned}$$

$$\begin{aligned}
(\Delta_\psi p)^2 &= \langle\psi, p^2\psi\rangle - (\langle\psi, p\psi\rangle)^2 \\
&= \tfrac{1}{2} - \left(\sqrt{2}\operatorname{Im} k\right)^2 \\
&= 2(\operatorname{Im} k)^2 + \tfrac{1}{2} - 2(\operatorname{Im} k)^2 \\
&= \tfrac{1}{2}
\end{aligned}$$

11. Thus, the standard deviations are

$$\Delta_\psi q = \Delta_\psi p = \tfrac{1}{\sqrt{2}},$$

so that

$$\Delta_\psi q \cdot \Delta_\psi p = \tfrac{1}{2},$$

the minimum allowed by the Heisenberg Uncertainty Principle!