A Coherent Assignment

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1. The cardinality $|C(k)_n|$ is the number of k-colourings of n, which is to say maps $f : n \to k$, into a set of colours the elements of n could be painted. This is just the number k^n, since each of n elements has k possible images under f, one for each "colour" it could be "painted", and these are independent.

2. The generating function for k-colourings is

$$|C(k)|(z) = \sum_{n \geqslant 0} \frac{|C(k)_n| z^n}{n!} = \sum_{n \geqslant 0} \frac{k^n z^n}{n!}$$

   This is just $\mathrm{e}^{kz} = \sum_{n \geqslant 0} \frac{(kz)^n}{n!}$.

3. Since the generating function $|C(k)|(z) = \mathrm{e}^{\mathrm{kz}}$, and the annihilation operator acts on power series by differentiation, we have

$$a|C(k)|(z) = \frac{\mathrm{d}}{\mathrm{d}z}|C(k)|(z) = \frac{\mathrm{d}}{\mathrm{d}z}\mathrm{e}^{kz} = k\mathrm{e}^{kz} = k|C(k)|(z).$$

   So in fact the generating function is an eigenvector (or eigenfunction) of this operator, with eigenvalue k, the number of colours in our colouring.

4. An eigenvector (or eigenfunction) of the annihilation operator (which is the $\frac{\mathrm{d}}{\mathrm{d}z}$ operator) will be a function for which there is some $\lambda \in \mathbb{C}$ for which $\frac{\mathrm{d}}{\mathrm{d}z}\psi(z) = \lambda\psi(z)$. Every eigenvector corresponds to a function satisfying this differential equation (as a formal power series). The solutions to this differential equation are all of the form $\psi(z) = \mathrm{e}^{\lambda z}$. In the case of the generating function $|C(k)|(z)$, this λ is k, a natural number (i.e. the cardinality of a particular finite set, the one from which we choose colours).

5. Recalling that the annihilation operator A acts on a structure type Ψ by producing $A\Psi$ which acts on a set S by putting a Ψ-structure on $S+1$, that is S with a new element added. In the situation where $\Psi = C(k)$, which is the structure of putting a k-colouring on a set, this $A\Psi$-structure is a k-colouring of $S+1$, which is the same thing (i.e. isomorphic as structure types) as the structure of a k-colouring on S together with a choice of colour from the set k (which had been assigned to the extra element 1 of $S+1$). This is a product of structure types, $kC(k)$, where a k-structure on a set S is a choice of a colour from k, together with the structure of being the empty set, and $C(k)$ is as usual. This gives an explicit isomorphism

$$AC(k) \cong kC(k)$$

6. Here, we notice that $\langle z^n, z^m \rangle = \langle (a^*)^n 1, z^m \rangle = \langle 1, (a)^n z^m \rangle$, and observe that since the a operator acts like the differentiation operator $\frac{\mathrm{d}}{\mathrm{d}z}$, if $n > m$ then this is just $\langle 1, 0 \rangle = 0$. On the other hand, if $m > n$ we could do the reverse, and have $\langle z^n, (a^*)^m 1 \rangle = \langle (a)^m z^n, 1 \rangle = \langle 0, 1 \rangle = 0$, again.

So the only way the inner product could be nonzero is if $n = m$. In this case, we have $\langle 1, (a)^n (a^*)^n 1\rangle = \langle 1, (a)^{n-1}(aa^*)(a^*)^{n-1}1\rangle$, but since $[a, a^*] = 1$, we have that $aa^* = a^*a + 1 = N + 1$, where N is the number operator which acts by $Nz^n = nz^n$. Thus, this last inner product is $\langle 1, (a)^{n-1}(N+1)z^{n-1}\rangle = n\langle 1, (a)^{n-1}(a^*)^{n-1}1\rangle$. Inductively, this gives us that $\langle z^n, z^n\rangle = n!\langle 1, 1\rangle$. This together with the fact that if $n \neq m$ the inner product is zero, tells us that $\langle z^n, z^m\rangle = \delta_{n,m} n!$

7. Any coherent state, as we saw in part 4, is of the form $\psi(z) = \mathrm{e}^{\lambda z}$, where λ is an eigenvalue. This is $\psi(z) = \sum_{n\geqslant 0} \frac{(kz)^n}{n!} = \sum_{n\geqslant 0} \frac{\lambda^n}{n!} z^n$. If we take the inner product $\langle \psi, \psi\rangle = \|\psi\|^2$, we use sesquilinearity and the fact that the z^n form an orthogonal set to see that $\langle \psi, \psi\rangle = \sum_{n\geqslant 0} (\frac{|\lambda|^n}{n!})^2 n! = \sum_{n\geqslant 0} \frac{|\lambda|^{2n}}{n!} = \mathrm{e}^{|\lambda|^2}$. So if this is $||\psi||^2$, then a normalized coherent state is
$$\frac{\psi}{\|\psi\|} = \frac{\mathrm{e}^{\lambda z}}{\sqrt{\mathrm{e}^{|\lambda|^2}}} = \frac{\mathrm{e}^{\lambda z}}{\mathrm{e}^{\frac{|\lambda|^2}{2}}} = \mathrm{e}^{\lambda(z - \frac{\overline{\lambda}}{2})}$$

8. We are using the expression above to refer to a normalized coherent state with eigenvalue λ (as an eigenvector of the annihilation operator), so the inner product which gives the expected value of momentum is, if we let $\psi = \mathrm{e}^{\lambda(z-\frac{\overline{\lambda}}{2})}$
$$\begin{aligned} \langle \psi, \boldsymbol{p}\psi\rangle &= \langle \psi, \tfrac{a-a^*}{\sqrt{2}\mathrm{i}}\psi\rangle \\ &= \tfrac{\langle \psi, a\psi\rangle - \langle a\psi, \psi\rangle}{\sqrt{2}\mathrm{i}} \\ &= \tfrac{\lambda\langle \psi,\psi\rangle - \overline{\lambda}\langle \psi,\psi\rangle}{\sqrt{2}\mathrm{i}} \\ &= \sqrt{2}\mathrm{Im}\lambda \end{aligned}$$
And on the other hand, the expected value of the position is:
$$\begin{aligned} \langle \psi, \boldsymbol{q}\psi\rangle &= \langle \psi, \tfrac{a+a^*}{\sqrt{2}}\psi\rangle \\ &= \tfrac{\langle \psi, a\psi\rangle + \langle a\psi, \psi\rangle}{\sqrt{2}} \\ &= \tfrac{\lambda\langle \psi,\psi\rangle + \overline{\lambda}\langle \psi,\psi\rangle}{\sqrt{2}} \\ &= \sqrt{2}\mathrm{Re}\lambda \end{aligned}$$

9. Similar reasoning to that in part 8 shows that the expected value of the momentum squared is

$$
\begin{aligned}
\langle\psi, \boldsymbol{p}^2\psi\rangle &= \left\langle \psi, \left(\tfrac{a-a^*}{\sqrt{2}\mathrm{i}}\right)^2 \psi \right\rangle \\
&= \tfrac{\langle\psi,(a^2-aa^*-a^*a+(a^*)^2)\psi\rangle}{-2} \\
&= \tfrac{\langle\psi,a^2\psi\rangle-\langle\psi,(a^*a+1)\psi\rangle-\langle\psi,a^*a\psi\rangle+\langle a^2\psi,\psi\rangle}{-2} \\
&= \tfrac{(\lambda^2-2\lambda\overline{\lambda}+\overline{\lambda}^2-1)\langle\psi,\psi\rangle}{-2} \\
&= 2(\mathrm{Im}\lambda)^2+\tfrac{1}{2}
\end{aligned}
$$

And likewise:

$$
\begin{aligned}
\langle\psi, \boldsymbol{q}^2\psi\rangle &= \left\langle \psi, \left(\tfrac{a+a^*}{\sqrt{2}}\right)^2 \psi \right\rangle \\
&= \tfrac{\langle\psi,(a^2+aa^*+a^*a+(a^*)^2)\psi\rangle}{2} \\
&= \tfrac{\langle\psi,a^2\psi\rangle+\langle\psi,(a^*a+1)\psi\rangle+\langle\psi,a^*a\psi\rangle+\langle a^2\psi,\psi\rangle}{-2} \\
&= \tfrac{(\lambda^2+2\lambda\overline{\lambda}+\overline{\lambda}^2-1)\langle\psi,\psi\rangle}{-2} \\
&= 2(\mathrm{Re}\lambda)^2+\tfrac{1}{2}
\end{aligned}
$$

10. Finding the variances from the results of 8 and 9, we find:

$$
\begin{aligned}
(\Delta_\psi \boldsymbol{p})^2 &= \langle\psi, \boldsymbol{p}^2\psi\rangle - (\langle\psi, \boldsymbol{p}\psi\rangle)^2 \\
&= 2(\mathrm{Im}\lambda)^2+\tfrac{1}{2}-2(\mathrm{Im}\lambda)^2 \\
&= \tfrac{1}{2}
\end{aligned}
$$

And again:

$$
\begin{aligned}
(\Delta_\psi \boldsymbol{q})^2 &= \langle\psi, \boldsymbol{q}^2\psi\rangle - (\langle\psi, \boldsymbol{q}\psi\rangle)^2 \\
&= 2(\mathrm{Re}\lambda)^2+\tfrac{1}{2}-2(\mathrm{Re}\lambda)^2 \\
&= \tfrac{1}{2}
\end{aligned}
$$

11. We saw that $(\Delta_\psi \boldsymbol{q})^2 = (\Delta_\psi \boldsymbol{p})^2 = \frac{1}{2}$. Thus, $\Delta_\psi \boldsymbol{q} = \Delta_\psi \boldsymbol{p} = \frac{1}{\sqrt{2}}$, which are equal and have product $\frac{1}{2}$.

12. If our coherent state is the generating function of the structure type $C(k)$, we have $\lambda = k \in \mathbb{N}$, so the expected position must be a natural number times $\sqrt{2}$, while the expected momentum must vanish. So, we can only categorify certain special coherent states! — so far, anyway.