Derek K. Wise
21 January 2004
Quantum Gravity Seminar
Homework #2 – Derangements

1.) $!n$ is the number of elements of S_n that are not in any stabilizer subgroup. Since $(S_n)_{\{i\}} \cong S_{n-1}$, we have $|(S_n)_{\{i\}}| = (n-1)!$. Similarly, $|(S_n)_{\{i,j\}}| = (n-2)!$ for $i \neq j$, and so on. Also, there are $\binom{n}{1}$ stabilizers of one element, $\binom{n}{2}$ stabilizers of two elements, and so on. Thus, by the inclusion-exclusion principle:

$$!n = |S_n| - \sum_i |(S_n)_{\{i\}}| + \sum_{i<j} |(S_n)_{\{i,j\}}| - \cdots + (-1)^n |(S_n)_{\{1,2,\ldots,n\}}|,$$

or

$$!n = n! - \binom{n}{1}(n-1)! + \binom{n}{2}(n-2)! - \cdots + (-1)^n (n-n)!,$$

or more concisely,

$$!n = \sum_{k=0}^{n} (-1)^k \binom{n}{k} (n-k)!.$$

2.) This can be simplified using the definition of $\binom{n}{k}$:

$$\begin{aligned} !n &= \sum_{k=0}^{n} (-1)^k \frac{n!}{k!(n-k)!}(n-k)! \\ &= n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \end{aligned}$$

3.) The probability of the coat-swapping being a derangement is

$$\frac{!n}{n!} = \sum_{k=0}^{n} \frac{(-1)^k}{k!}$$

Hence,

$$\lim_{n\to\infty} \frac{!n}{n!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}.$$

4.) **T**heorem: For $n > 0$, $!n$ is the closest integer to $n!e^{-1}$.

Proof: Since we know $!n$ is an integer, we only need to show that the absolute difference between $!n$ and $n!e^{-1}$ is less than a half.

$$\begin{aligned}\left|\frac{n!}{e} - !n\right| &= \left| n! \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} - n! \sum_{k=0}^{n} \frac{(-1)^k}{k!} \right| \\ &= n! \left| \sum_{k=n+1}^{\infty} \frac{(-1)^k}{k!} \right| \\ &< n! \left| \frac{1}{(n+1)!} \right| \\ &= \frac{1}{n+1}\end{aligned}$$

Here we have used the fact that an alternating series whose terms are decreasing in magnitude is strictly dominated by its first term. This proves the theorem for $n \geq 2$. Since the case $n = 1$ is easily verified ($!1 = 0$, and $1!/e \approx .37$), we are done. ♠

5.) To permute the elements of a set, we decide which elements will remain fixed under the permutation and then we *derange* the rest. That is, we split the set into two (possibly empty) parts, putting the vacuous structure"being a finite set" on the first part, and deranging the second part. Thus,

$$P \cong E^Z D.$$

6.) Decategorifying, we get the following equation between generating functions:

$$|P|(z) = e^z |D|(z).$$

If we rearrange this and use the fact that $|P|(z)$ is just a geometric series, we get:

$$|D|(z) = e^{-z}|P|(z) = \frac{e^{-z}}{1-z}$$

7.)

$$\begin{aligned}(1-z)\frac{d}{dz}|D|(z) &= (1-z)\frac{d}{dz}\frac{e^{-z}}{1-z} \\ &= (1-z)\left(\frac{e^{-z}}{(1-z)^2} - \frac{e^{-z}}{1-z}\right) \\ &= |D|(z) - e^{-z}.\end{aligned}$$

8.) We now apply the result of part 7 to the power series representation of $|D|$, given by

$$|D|(z) = \sum_{n=0}^{\infty} \frac{!n}{n!} z^n.$$

For the left hand side we get:

$$\begin{aligned}(1-z)\frac{d}{dz}|D|(z) &= (1-z)\frac{d}{dz}\sum_{n=0}^{\infty}\frac{!n}{n!}z^n. \\ &= (1-z)\sum_{n=1}^{\infty}\frac{!n}{(n-1)!}z^{n-1} \\ &= \sum_{n=1}^{\infty}\frac{!n}{(n-1)!}z^{n-1} - \sum_{n=1}^{\infty}\frac{!n}{(n-1)!}z^n \\ &= \sum_{n=0}^{\infty}\frac{!(n+1)}{n!}z^n - \sum_{n=1}^{\infty}\frac{!n}{(n-1)!}z^n \\ &= \frac{!1}{0!}z^0 + \sum_{n=1}^{\infty}\left[\frac{!(n+1)}{n!} - \frac{!n}{(n-1)!}\right]z^n \\ &= \sum_{n=1}^{\infty}\left[\frac{!(n+1)}{n!} - \frac{!n}{(n-1)!}\right]z^n.\end{aligned}$$

And for the right hand side:

$$|D|(z) - e^{-z} = \sum_{n=0}^{\infty}\left[\frac{!n}{n!} - \frac{(-1)^n}{n!}\right]z^n$$

or, since the first term is zero,

$$|D|(z) - e^{-z} = \sum_{n=1}^{\infty}\left[\frac{!n}{n!} - \frac{(-1)^n}{n!}\right]z^n.$$

Equating corresponding coefficients in our two power series representations of $|D|(z)$ we see that for $n \geq 1$,

$$\frac{!(n+1)}{n!} - \frac{!n}{(n-1)!} = \frac{!n}{n!} - \frac{(-1)^n}{n!}.$$

Multiplying this by $n!$, we get the desired result:

$$!(n+1) - n\ !n = !n - (-1)^n$$

or,

$$!(n+1) = (n+1)\ !n + (-1)^{n+1}.$$

So far, we have only shown this is true for $n \geq 1$, but it is easily verified to hold also when $n = 0$.

9.) I'm not sure the first 6 will be sufficient to convince me that this stuff is *really* cool, so I'll do the first 7:

$!0 = 1$		.	
$!1 = 1(1) - 1 = 0$	$1!/e =$	$.367\ldots$	≈ 0
$!2 = 2(0) + 1 = 1$	$2!/e =$	$.735\ldots$	≈ 1
$!3 = 3(1) - 1 = 2$	$3!/e =$	$2.207\ldots$	≈ 2
$!4 = 4(2) + 1 = 9$	$4!/e =$	$8.829\ldots$	≈ 9
$!5 = 5(9) - 1 = 44$	$5!/e =$	$44.15\ldots$	≈ 44
$!6 = 6(44) + 1 = 265$	$6!/e =$	$264.87\ldots$	≈ 265
$!7 = 7(265) - 1 = 1854$	$7!/e =$	$1854.11\ldots$	≈ 1854

Hey, that's really cool!