Derangements

Miguel Carrión Álvarez

1,2. *The principle of inclusion-exclusion.*

The number of permutations of n elements fixing k of them is equal to the number of ways to choose the k fixed elements, times the number of permutations of the remaining $n-k$ elements; that is,

$$\binom{n}{k}(n-k)!.$$

By the principle of inclusion-exclusion,

$$!n = \sum_{0\le k\le n} (-1)^k \binom{n}{k}(n-k)!$$

Since $\binom{n}{k} = \frac{n!}{k!(n-k)!}$, this implies

$$!n = n! \sum_{0\le k\le n} \frac{(-1)^k}{k!}$$

3. *Asymptotics.*

The mathematicians are effectively choosing one permutation of their coats at random, all $n!$ permutations being equally likely. The number of cases in which no mathematician picks their own coat is $!n$, the number of derangements. Since

$$\frac{!n}{n!} = \sum_{0\le k\le n} \frac{(-1)^k}{k!},$$

it follows that

$$\lim_{n\to\infty} \frac{!n}{n!} = \sum_{k\ge 0} \frac{(-1)^k}{k!} = e^{-1}.$$

4. *Approximating* $!n$.

$$\frac{n!}{e} - !n = n! \sum_{k\ge 0} \frac{(-1)^k}{k!} - n! \sum_{0\le k\le n} \frac{(-1)^k}{k!} = n! \sum_{k>n} \frac{(-1)^k}{k!}.$$

Since an alternating sum of decreasing elements is smaller than its first element in absolute value,

$$\left|\frac{n!}{e} - !n\right| < \frac{n!}{(n+1)!} = \frac{1}{n+1} \le 1/2$$

if $n \ge 0$.

5. *Definition of* D.

Since a derangement is a permutation without fixed points, it is clear that a permutation of a set S divides S into two disjoint subsets, the set S_1 of fixed points and its complement S_2, the largest subset of S on which the permutation is a derangement. Conversely, dividing S into two subsets, S_1 and S_2, and deranging S_2, is a permutation of S. The structure type "being a finite set" is such that there is exactly one way to put it on any set, and so its generating function is e^z. Accordingly, we denote the structure type by e^Z. By the definition of multiplication of structure types, it follows that

$$P = e^Z D,$$

where P is the structure type of permutations and D that of derangements.

6. *Generating function.*

The number of permutations of a set S is the same as the number of total orders on it (more precisely, the set of total orders on n is an S_n-torsor). Hence, the generating function of P is the same as that for total orders: $|P|(z) = 1 + z + z^2 + \cdots + z^n + \cdots = (1-z)^{-1}$. Therefore,

$$|P|(z) = e^z |D|(z)$$

implies

$$\frac{1}{1-z} = e^z |D|(z)$$

so

$$|D(z)| = \frac{e^{-z}}{1-z}.$$

7. *Differential equation for $|D|(z)$.*

Differentiating

$$(1-z)|D|(z) = e^{-z},$$

one obtains

$$-|D|(z) + (1-z)\frac{d}{dz}|D|(z) = -e^{-z},$$

so

$$(1-z)\frac{d}{dz}|D|(z) = |D|(z) - e^{-z}.$$

8. *Recurrence relation for $!n$.*

If

$$|D|(z) = \sum_{n\geq 0} \frac{!n}{n!} z^n,$$

then

$$\frac{d}{dz}|D|(z) = \sum_{n\geq 0} \frac{!(n+1)}{n!} z^n.$$

The differential equation from part 6) then implies

$$(1-z)\sum_{n\geq 0} \frac{!(n+1)}{n!} z^n = \sum_{n\geq 0} \frac{!n}{n!} z^n - \sum_{n\geq 0} \frac{(-1)^n}{n!} z^n$$
$$\sum_{n\geq 0} \frac{!(n+1)}{n!} z^n - \sum_{n\geq 0} n\frac{!n}{n!} z^n = \sum_{n\geq 0} \frac{!n}{n!} z^n - \sum_{n\geq 0} \frac{(-1)^n}{n!} z^n$$
$$\sum_{n\geq 0} \frac{!(n+1)}{n!} z^n = \sum_{n\geq 0} (n+1)\frac{!n}{n!} z^n - \sum_{n\geq 0} \frac{(-1)^n}{n!} z^n$$

We then get the recurrence relation

$$!(n+1) = (n+1)!n - (-1)^n.$$

9. *Calculating derangements.*

n	$n!$	$n!/e$	$!n$
0	1	0.36788	1
1	$1 \cdot 1 = 1$	0.36788	$1 \cdot 1 - 1 = 0$
2	$2 \cdot 1 = 2$	0.73576	$2 \cdot 0 + 1 = 1$
3	$3 \cdot 2 = 6$	2.2073	$3 \cdot 1 - 1 = 2$
4	$4 \cdot 6 = 24$	8.8291	$4 \cdot 2 + 1 = 9$
5	$5 \cdot 24 = 120$	44.146	$5 \cdot 9 - 1 = 44$
6	$6 \cdot 120 = 720$	264.87	$6 \cdot 44 + 1 = 265$