QUANTUM GRAVITY HOMEWORK 3

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1. By direct analogy with the number of ways to write down a finite number as a sum of 1's and 2's,

$$p(z) = \sum_{n\geq 0} p_n z^n = \frac{1}{1-z} \cdot \frac{1}{1-z^5} \cdot \frac{1}{1-z^{10}}$$

2. To find the number of different ways to make change for \$10 using only pennies, nickels, and dimes, we use the following Mathematica code:

$$\mathsf{In[6]}{:=}\, p[z\_] := \text{Series}\left[\tfrac{1}{1-z}\tfrac{1}{1-z^5}\tfrac{1}{1-z^{10}}, \{z, 0, 1000\}\right]$$

$$\mathsf{In[7]}{:=}\, \text{SeriesCoefficient}\,[p[z], 1000]$$

$$\mathsf{Out[7]}{=}\, 10201$$

3. If p_n is the number of ways to write n as a sum of positive natural numbers chosen from some set $S \subseteq \mathbb{N}^+$,then

$$p(z) = \sum_{n\geq 0} p_n z^n = \prod_{s\in S} \frac{1}{1-z^s}$$

4. Let $S = \{s_1, s_2, \dots, s_k\} \subseteq \mathbb{N}^+$. A P-structure consists of chopping a set A into $k = |S|$ parts, totally ordering each, then chopping the first into blocks of length s_1, the second into blocks of length s_2, etc.

   Now we obtain the isomorphism

$$P \cong \frac{1}{1-Z^{s_1}} \cdot \frac{1}{1-Z^{s_2}} \cdots \frac{1}{1-Z^{s_k}},$$

   because each term $(1-Z^{s_j})^{-1}$ corresponds to the structure type of totally ordering a set and chopping it into blocks of length s_j.

5. If q_n is the number of ways to write n as a sum of positive numbers (chosen from $S \subseteq \mathbb{N}^+$), where each number can be used only once, then

$$q_n = \sum_{n\geq 0} q_n z^n = \prod_{s\in S} (1+z^s).$$

   The justification for this comes by decategorifying 6.

6. Let $S = \{s_1, s_2, \dots, s_k\} \subseteq \mathbb{N}^+$, where all the s_j are distinct. A Q-structure on a set A consists of chopping A into $k = |S|$ parts. The first part is either empty or else it gets totally ordered and endowed with the structure of "being an s_1-element set". The second part is treated similarly, etc. This leads to the isomorphism
$$Q \cong (1 + Z^{s_1})\,(1 + Z^{s_2})\dots(1 + Z^{s_k})\,.$$

7. The number of ways of writing n as a sum of odd numbers is
$$\begin{aligned}\prod_{n\geq 0}\frac{1}{1-z^{2n+1}} &= \frac{1}{1-z^1}\cdot\frac{1}{1-z^3}\cdot\frac{1}{1-z^5}\cdots\\ &= \frac{1-z^2}{1-z}\cdot\frac{1-z^4}{1-z^2}\cdot\frac{1-z^6}{1-z^3}\cdot\frac{1-z^8}{1-z^4}\cdots\\ &= (1+z)\cdot(1+z^2)\cdot(1+z^3)\cdot(1+z^4)\\ &= \prod_{n\geq 0}(1+z^n),\end{aligned}$$
which is the number of ways of writing n as a sum of distinct numbers.