Counting Partitions

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Abstract

𝔅eing a 𝔐oſt 𝔐errie 𝔈xploration of the 𝔠ounting of 𝔓artitionſ, or: 𝔄n 𝔄ccount of the 𝔐anp and 𝔖everal 𝔚apſ to 𝔐ake 𝔠hange in 𝔣oreign 𝔏andſ whoſe 𝔠oinage be of the 𝔐oſt 𝔈ccentric 𝔙alueſ... 𝔍ncluding a 𝔠ategoricall 𝔇emonſtration of a 𝔇iſtinctive 𝔒ddity at 𝔑o 𝔈xtra 𝔠harge

1. We can define a structure type $P_{\{1,5,10\}}$ which acts on an n-set N by putting an order on S and then cutting it into 0 or more blocks of size 1, then cutting the remainder into 0 or more blocks of size 5, then cutting the remainder into blocks of size 10. That is, a $P_{\{1,5,10\}}$ structure is equivalent to the structure formed by first taking an ordered set of (ordered) sets of size 1, then an ordered set of ordered sets of size 5, then an ordered set of ordered sets of size 10. The generating function for structure consisting of an ordered set of ordered sets of size n is $1 + z^n + z^{2n} + ... = \frac{1}{1-z^n}$. We have a product of such structures, so the generating function $p(z)$ is a product of terms of this form, with $n = 1, 5, 10$:

$$p(z) = \sum_{n \geqslant 0} p_n z^n = \left(\frac{1}{1-z}\right)\left(\frac{1}{1-z^5}\right)\left(\frac{1}{1-z^{10}}\right)$$

2. To make change for ten dollars in pennies, nickels, and dimes is to express 1000 (cents) as a sum of units of sizes 1, 5, and 10. This is just p_{1000}, which is given by the above as the coefficient of z^{1000} in $p(z)$. Computers being what they are, this is easily found to be[1]:

$$[z^{1000}]\left(\frac{1}{1-z}\right)\left(\frac{1}{1-z^5}\right)\left(\frac{1}{1-z^{10}}\right) = 10201$$

3. By analogy with the answer to (1), this will be a product over values $n \in S$ of geometric series:

$$p(z) = \sum_{n \geqslant 0} p_n z^n = \prod_{s_i \in S}\left(\frac{1}{1-z^{s_i}}\right)$$

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1. We could note that if we were stuck on a desert island with no electricity, we could still calculate this without much trouble, since it amounts to the number of ways to pick a number d of dimes between 0 and 100 (there are 101 ways), and for each of these, pick a way to write some portion of the remaining quantity in nickels, that is, some number n_{ickel} between 0 and $200 - 2d$ (the number of nickels needed to make up the rest) since automatically the rest will be pennies. This number is:

$$\sum_{d=0}^{100} (201 - 2d) = (201)(101) - 2\left(\frac{100^2 + 100}{2}\right) = 20301 - 10100 = 10201$$

But, on the other hand, it's worth remarking that this convenient shortcut is essentially a result of the fact that 5 divides 10, and 1 divides both of them - and all three numbers divide 1000 - greatly simplifying the formula for the number of ways to choose boundary points between groups of blocks of these sizes. In general, if we have some other set S of coin values and total value n which we wish to divide, the first method - calculating coefficient of z^n - will be the better one.

4. We can define a structure-type P (or, to clearly distinguish it, P_S) for which a P_S-structure on an n-element set N is a total ordering of N, and a division of this ordered set into blocks of sizes $s_i \in S$, in decreasing order of size (for instance - though taking the s_i in any specified order on $\mathbb{N}^+$ would do equally well). A P_S-structure is equivalent to a partition of n by elements of S, together with an order, hence the ordinary generating function will have coefficients giving the number of such partitions. Now, a P_S-structure on N consists of a collection of blocks, so we can think of it as a product of structures; first, cut N into two parts, and put the structure of being-an-ordered-set-of-ordered-sets-of-size-s_1 on the first part, and on the second part put a structure which consists of cutting a set in two parts, and putting the structure of ordered-set-of-ordered-sets-of-size-s_2 on the first part, and... And so on, through all s_i. This gives P_S as a product of structures, since the recursive construction naturally gives an order on all the sub-structures, each of which naturally puts an order on the elements of N which it uses, and this product therefore gives an ordered collection of ordered blocks of the appropriate sizes.

   This construction of P_S as a product of structure-types can be written:
   $$P_S = \prod_{s_i \in S} (1 + Z^{s_i} + Z^{2s_i} + ...) = \prod_{s_i \in S} \frac{1}{1 - Z^{s_i}}$$
   (Where the "geometric series" has been written in the suggestive notation as $\frac{1}{1 - Z^{s_i}}$, for whatever that means). Decategorifying this gives the generating function above.

5. This is similar to the case in (3), but instead of the geometric series associated, we only have a two term polynomial for each $s_i \in S$, since it either appears once, or not at all, rather than any number of times. The generating function is:
   $$q(z) = \sum_{n \geqslant 0} q_n z^n = \prod_{s_i \in S} (1 + z^{s_i})$$

6. The construction here is just as in (4), except that the basic structures from which Q is recursively constructed are not ordered-sets-of-ordered-sets-of-size-s_i, but rather are structures defined as: "either the empty set, or an ordered set of size s_i". So rather than the geometric series $(1 + Z^{s_i} + Z^{2s_i} + \cdots)$, these structures can be written $(1 + Z^{s_i})$. So by the same reasoning as in (4), we get that
   $$Q_S = \prod_{s_i \in S} (1 + Z^{s_i} + Z^{2s_i} + ...) = \prod_{s_i \in S} \frac{1}{1 - Z^{s_i}}$$

7. We can show this either at the level of generating functions, or directly in the categorified setting:

   i. The number of ways of writing n as a sum of distinct numbers is the situation from (5), in the case where $S = \mathbb{N}^+$: we are finding a partition of n as a sum of positive numbers, where each appears at most once. So from (5), the generating function $q(z)$, whose coefficients q_n are the number of ways of doing this for n, is:
      $$q(z) = \prod_{s_i \in S} (1 + z^{s_i})$$
      Now, if we take this expression and use the fact that $1 + z^n = \frac{1 - z^{2n}}{1 - z^n}$, we have:
      $$\prod_{n \geq 1} (1 + z^n) = \prod_{n \geq 1} \frac{1 - z^{2n}}{1 - z^n}$$
      $$= \prod_{n \text{ odd}} \frac{1}{1 - z^n}$$

      Here the second equality comes about because we can cancel all the denominators with even powers of z with the terms in which these appear as numerators. All numerators are cancelled in this way since every even positive integer $2n$ will also appear as a positive integer. All even-power denominators cancel this way, and so only and all odd-power denominators remain.

But now, this expression is actually a special case of the form from (3), where the set S is the set of odd numbers. So the interpretation is that this is also the generating function $p(z)$ whose coefficients are the number of partitions of n with odd parts (which may appear any number of times). Since these are equal as generating functions, the coefficients are all equal, hence for every $n \in \mathbb{N}^+$, the number of ways of writing n as a sum of distinct numbers is the same as the number of ways of writing n as a sum of odd numbers.

ii. To show this more directly, we want to construct some sort of bijection between the set of partitions of k by odd numbers and the set of partitions of k by distinct numbers. Categorifying the generating-function equations above, we observe that the structure-type $Q_{\mathbb{N}+}$ of partitions of an ordered finite set with distinct-size parts can be thought of in the following way. The sub-structure "either nothing or exactly one ordered set of size n" from which the structure was build recursively (by taking the product over all n), is the same as the structure "ordered sets of ordered sets of size n, exclusive of sets with two or more elements":

$$1 + Z^n = \frac{1}{1 - Z^n} - \frac{Z^{2n}}{1 - Z^n}$$

Since:

$$\frac{1}{1 - Z^n} = 1 + Z^n + Z^{2n}(1 + Z^n + \cdots) = 1 + Z^n + \frac{Z^{2n}}{1 - Z^n}$$

But this means that a block in an arbitrary partition can be thought of as: either the empty set, or a single part of size n, or a part of size $2n$ together with an arbitrary block with parts of size n. To restrict to partitions having distinct sizes, we must remove all those with blocks of the third type. Since there is a natural isomorphism between partitions with parts of the third type and those with one or parts of even size (given by taking the part of size $2n$ in the description from the n-block and putting it in the $2n$-block) this is equivalent to removing ALL partitions having parts of size $2n$ for any n. That is, removing all partitions having any parts of even size. This leaves only partitions with blocks of odd size.