**Partitions**

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**1,2.** *Cents, nickels and dimes.*

To change \$10 into cents, nickels and dimes is the same as expressing 1000 as a sum of 1's, 5's and 10's. By analogy with expressing a number as a sum of 1's and 2's, the generating function of the number of ways of decomposing $n$ into 1's, 5's and 10's, denoted $p_n$, is

$$P(z) = \sum_{n\geq 0} p_n z^n = \frac{1}{1-z}\frac{1}{1-z^5}\frac{1}{1-z^{10}}.$$

Using Mathematica, we see that $p_{1000} = 10201$.

**3.** *Partitions into elements of $S \subseteq \mathbf{N}$.*

$$P_S(z) = \prod_{s\in S} \frac{1}{1-z^s}$$

**4.** *Categorification.*

If $S$ is the empty set, call an $S$-structure on a set the structure of "being the empty set". This structure type is also called 1.

If $S = \{s\}$ is a singleton, putting an $S$-structure on a set is the same as ordering the set and breaking it into blocks of $s$ elements. This structure type is called

$$\frac{1}{1-Z^s}.$$

Denote the smallest element of $S$ by $s$, and let $S' = S - \{s\}$. Then, an $S$-structure on a set $T$ is equivalent to dividing $T$ into two parts, putting an $\{s\}$-structure on the first and an $S'$-structure on the second. Then,

$$P_S = \frac{1}{1-Z^s} \times P_{S'}.$$

**5,6.** *Pauli partitions.*

If $n > 0$, call a $Q_n$-structure on a set "being empty or being an ordered $n$-element set". This structure type satisfies

$$Q_n \simeq 1 + Z^n.$$

It follows that

$$Q_S(z) = \prod_{s\in S}(1+z^s).$$

**7.** Odd = distinct.

The generating function of a partition by odd numbers is

$$O(z) = \prod_{n\geq 1} \frac{1}{1-z^{2n-1}}$$

while the generating function of a partition by distinct numbers is

$$D(z) = \prod_{n\geq 1}(1+z^n).$$

Observe that

$$D(z) = \prod_{n\geq 1}(1+z^n) = \prod_{n\geq 1}\frac{1-z^{2n}}{1-z^n}.$$

Since all the terms with even exponents appear exactly once in the numerator and denominator, and all the terms with odd exponents appear exactly once in the denominator, we have

$$D(z) = O(z).$$