13 Jan 2004

ISO BETWEEN STRUCTURE TYPES $\quad F \cong Z^2F + ZF + 1$

$\downarrow$

EQ BETWEEN GEN FUNCTIONS $\quad |F| = z^2|F| + z|F| + 1 \quad \underset{\longleftarrow}{\xrightarrow{\text{power series}}} \quad$ RECURRENCE $\quad |F_n| = |F_{n-2}| + |F_{n-1}|$

$n > 1$
$|F_0| = 1$
$|F_1| = |F_0|$

$\downarrow$

SOLVE FOR GEN FUNCTION $\quad |F| = \dfrac{1}{1 - z - z^2}$

this problem is best solved by going the other way around the diagram — i.e. via generating functions.

$\downarrow$ $\boxed{\text{solving a recurrence}}$

PARTIAL FRAC $\quad |F| = \dfrac{A}{z - z_0} + \dfrac{B}{z - z_1} \quad \xrightarrow[\text{(expand using geom. series)}]{\text{power series}} \quad |F_n| = A \ldots + B \ldots$

The Harmonic Oscillator with Many Degrees of Freedom:

For the harmonic oscillator with one degree of freedom, we got a Hilbert space containing $\mathbb{C}[z]$ with operators

$$a, a^* : \mathbb{C}[z] \longrightarrow \mathbb{C}[z]$$

given by

$$(a\psi)(z) = \psi'(z)$$
$$(a^*\psi)(z) = z\psi(z)$$

$\mathbb{C}[z]$ gets an inner product by demanding

$$\langle a^*\psi, \varphi\rangle = \langle \psi, a\varphi\rangle \qquad \forall \psi, \varphi \in \mathbb{C}[z]$$

$$\& \quad \|1\| = 1$$

(The details of this are in a forthcoming homework assignment.)

Then we form a Hilbert space by completing $\mathbb{C}[z]$ in this inner product — let's call this Hilbert space <u>Fock Space</u> $K[z]$. Note $\mathbb{C}[z]$ is a dense subset of $K[z]$, which is contained in $\mathbb{C}[[z]]$:

$$\mathbb{C}[z] \underset{\text{dense}}{\subset} K[z] \subset \mathbb{C}[[z]]$$

$K[z]$ will consist of $\sum_{n\geq 0} a_n z^n \in \mathbb{C}[[z]]$ where the a_n satisfy some condition like $\sum_{n\geq 0} n!\,|a_n|^2 < \infty$ ensuring the Hilbert space norm is finite.

Now let's try to mimic this for a harmonic oscillator with many degrees of freedom. Now we consider N variables $z_1, \ldots, z_N$ & N annihilation/creation operators:

$$a_i, a_i^* : \mathbb{C}[z_1, \ldots, z_N] \longrightarrow \mathbb{C}[z_1, \ldots, z_N]$$

where $1 \leq i \leq N$,

with

$$a_i \psi = \frac{\partial}{\partial z_i} \psi$$

$$a_i^* \psi = z_i \psi$$

Note:

$$[a_i, a_j] = \frac{\partial}{\partial z_i}\frac{\partial}{\partial z_j} - \frac{\partial}{\partial z_j}\frac{\partial}{\partial z_i} = 0$$

(Note: these are being applied to polynomials)

$$[a_i, a_j^*] = \delta_{ij} 1$$

$$[a_i^*, a_j^*] = z_i z_j - z_j z_i = 0$$

These are the a, a^* version of the canonical commutation relations. |

The idea now will be to generalize what we did along these lines. For example, we could have

$$H = \sum_{i=1}^{N} (a_i^* a_i + \tfrac{1}{2})$$

This works for an isotropic harmonic oscillator. More generally,

$$H = \sum_{i=1}^{N} E_i \, (a_i^* a_i + \tfrac{1}{2})$$

where E_i depends on the spring constant of the oscillator in the ith direction. Or: E_i is the energy of a quantum of the ith kind.

For example:

$\mathbb{C}[z_1, z_2]$ ~~with $E_1 = 1$, $E_2 = 2$~~

$$H1 = \sum_{i=1}^{2} E_i \left(z_i \frac{\partial}{\partial z_i} + \frac{1}{2}\right) 1 = E_1 \cdot \frac{1}{2} + E_2 \cdot \frac{1}{2}$$

$$H z_1 = \sum_{i=1}^{2} E_i \left(z_i \frac{\partial}{\partial z_i} + \frac{1}{2}\right) z_1 = \left(E_1 \cdot \frac{3}{2} + E_2 \cdot \frac{1}{2}\right) z_1$$

$$H z_2 = \sum_{i=1}^{2} E_i \left(z_i \frac{\partial}{\partial z_i} + \frac{1}{2}\right) z_2 = \left(E_1 \cdot \frac{1}{2} + E_2 \cdot \frac{3}{2}\right) z_2$$

$$H z_1 z_2 = \sum_{i=1}^{2} E_i \left(z_i \frac{\partial}{\partial z_i} + \frac{1}{2}\right) z_1 z_2 = \left(E_1 \cdot \frac{3}{2} + E_2 \cdot \frac{3}{2}\right) z_1 z_2$$

or generally:

$$H z_1^n z_2^m = \sum_{i=1}^{2} E_i \left(z_i \frac{\partial}{\partial z_i} + \frac{1}{2}\right) z_1^n z_2^m = \left[E_1 \left(n + \frac{1}{2}\right) + E_2 \left(m + \frac{1}{2}\right)\right] z_1^n z_2^m$$

since

$$z_1 \frac{\partial}{\partial z_1} z_1^n z_2^m = n z_1^n z_2^m$$

$$z_2 \frac{\partial}{\partial z_2} z_1^n z_2^m = m z_1^n z_2^m$$

this is a special case of...

<u>Euler's Thm on Homogeneous Funcions:</u>

If $f : \mathbb{R} \to \mathbb{R}$ is homogeneous of degree n:

$$f(cx) = c^n f(x)$$

then

$$x \frac{d}{dx} f = n f.$$

Moral: ground state energy is

$$\sum_{i=1}^{N} E_i \frac{1}{2}$$ & adding a quantum of type i increases the energy by E_i.

We can also subtract the vacuum energy & define a new Hamiltonian:

$$H_0 = \sum_{i=1}^{N} E_i a_i a_i^*$$

Now if $N=2$, $E_1 = 1$, $E_2 = 2$, how many states have energy E?

$$H_0 1 = 0.$$
$$H_0 z_1 = E_1$$
$$H_0 z_2 = E_2$$
$$H_0 z_1 z_2 = E_1 + E_2$$
$$H_0 z_1^n z_2^m = E_1 n + E_2 m$$

Energy	# of States	States	
$E = 0$	1	$(n,m) = (0,0)$	
1	1	$(1,0)$	•
2	2	$(2,0), (0,1)$	•|• , ••
3	2	$(3,0), (1,1)$	•|•|• , •|••
4	3	$(4,0), (2,1), (0,2)$	•|•|•|• , •|•|•• , ••|••
⋮	⋮	⋮	

← This sequence is just 1, 1, 2, 2, 3, 3, 4, 4, ...

15 Jan 2004

Last time we were studying

$$H_0 = \sum_{i=1}^{2} E_i a_i^* a_i$$

where $E_1 = 1$ and $E_2 = 2$, i.e. a system w/ one kind of particle of energy 1, one kind of energy 2.

We counted the states of energy E:

		# states
$E = 0$		1
$E = 1$	1	1
$E = 2$	$1+1,\ 2$	2
$E = 3$	$1+1+1,\ 2+1$	2
$E = 4$	$1+1+1+1,\ 2+1+1,\ 2+2$	3

& saw it goes up one every other time. Can we find a generating fn for this sequence g_n:

$$|G|(z) = \sum_{n=0}^{\infty} g_n z^n \ ?$$

(The $n!$ ~~was~~ is cancelled by $n!$ ways of choosing a total ordering – so we are making the assumption ~~that~~ our structure involves ordering the set first)

Note: G is like the fibonacci str. type F except now all blocks of size 2 must come first:

$$F_3 = \left\{ \begin{array}{ccc} \underbrace{\begin{array}{cc} 1|2|3 & 1\ 2|3 \\ 2|1|3 & 2\ 1|3 \\ 1|3|2 & 1\ 3|2 \\ 3|1|2 & 3\ 1|2 \\ 2|3|1 & 2\ 3|1 \\ 3|2|1 & 3\ 2|1 \end{array}}_{(G_3)} & \begin{array}{c} 1|2\ 3 \\ 2|1\ 3 \\ 1|3\ 2 \\ 3|1\ 2 \\ 2|3\ 1 \\ 3|2\ 1 \end{array} \end{array} \right\}$$

$|F_3| = 3!\, f_3$

of these, $3! \cdot 2 = 3!\, g_3$ are G-structures.

To put a G-str. on S we totally order S & chop it into parts; put the str. "being an even set" on the first part & put the structure "being a finite set" on the second part.

Or Better:

Chop S into two parts; put the str. "being a totally ordered even set" on the first part; put the str. "being a totally ordered finite set" on the second part.

Now G is a product of 2 structure types:

"Being a totally ordered set":

$$1 + Z + Z^2 + Z^3 + \cdots = \frac{1}{1-Z}$$

"Being a totally ordered even set":

$$1 + Z^2 + Z^4 + \cdots = \frac{1}{1-Z^2}$$

So:

$$G = \frac{1}{1-Z^2} \, \frac{1}{1-Z}$$

and we get the generating function

$$\begin{aligned} |G|(z) &= \frac{1}{1-z} \, \frac{1}{1-z^2} \\ &= (1 + z + z^2 + z^3 + \cdots)(1 + z^2 + z^4 + \cdots) \\ &= 1 + z + 2z^2 + 2z^3 + 3z^4 + 3z^5 \end{aligned}$$

note:

$$= (z^0 + z^1 + z^{1+1} + z^{1+1+1} + \cdots) \cdot (z^0 + z^2 + z^{2+2} + z^{2+2+2})$$

compare this to the table of energies & states

We could write

$$G = \frac{1}{1 - Z - Z^2 + Z^3} \qquad (?!)$$

but we don't understand this as an equation of str. types.
Let's dive in and try anyway

$$(1 - Z - Z^2 + Z^3)\, G = 1$$

$$G \cong GZ + GZ^2 \underbrace{- GZ^3}_{\text{huh?!}} \;?$$

— this must be the part that eliminates the F-structures that aren't G-structures, since: $F = FZ + FZ^2$.

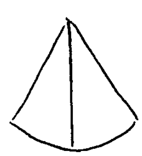
$$G + GZ^3 \cong GZ + GZ^3 \;?$$

does such an isomorphism exist?

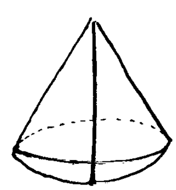
Back to harmonic oscillators:

We've seen if we have a harmonic oscillator that can wiggle in N different directions & we quantize it, we get as our Hilbert space (some Hilbert space completion of)

$$\mathbb{C}[z_1, \ldots, z_N]$$

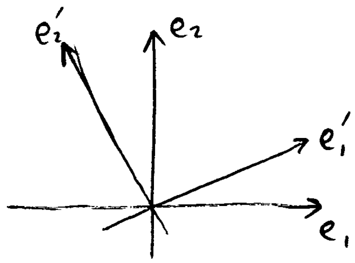


$N = 1$



$N = 2$

But, our oscillator might not come with N favorite directions to wiggle in:



So, instead of starting with a set of directions $S = \{1, \ldots, N\}$ we could start w. an n-dimensional vector space V, & form the symmetric algebra on V:

$$SV = TV / \langle x \otimes y - y \otimes x \rangle$$

i.e. TV, the <u>tensor</u> <u>algebra</u> on V is the free associative algebra on V, while SV is the free commutative (assoc.) algebra on V

So:

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, "S \mapsto \mathbb{C}S"] \arrow[dr, "S \mapsto \mathbb{C}[S]"'] & \mathrm{Vect}_{\mathbb{C}} \arrow[d, "V \mapsto SV"] \\
& \mathrm{CommAlg}_{\mathbb{C}}
\end{tikzcd}$$

where $\mathbb{C}[S]$ is the polynomial algebra on (variables indexed by) S & $\mathbb{C}S$ is the vector space whose basis is S, i.e. $x \in \mathbb{C}S$ is a formal linear combination of elements of S. These are examples of "free functors."