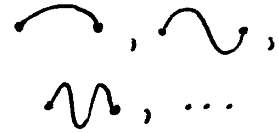
20 Jan 2004

Last time we had:

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, "S \mapsto \mathbb{C}S"] \arrow[dr, "S \mapsto \mathbb{C}[S]"'] & \mathrm{Vect}_{\mathbb{C}} \arrow[d, "V \mapsto SV"] \\
 & \mathrm{CommAlg}_{\mathbb{C}}
\end{tikzcd}$$

e.g. A string: $S = \{$  $, \ldots \}$

Given a set S of "vibrational modes" we can form the complex vector space $\mathbb{C}S$ with basis S; it's just the classical phase space. E.g. if $S = 1$, $\mathbb{C}S = \mathbb{C}$ is the classical phase space for the harmonic oscillator with 1 degree of freedom:

$$\mathbb{C} \ni z = q + ip$$

describes position & momentum of the oscillator. If $S = n$, $\mathbb{C}S = \mathbb{C}^n$ & we get $(z_1, \ldots, z_n) = (q_1 + ip_1, \ldots, q_n + ip_n)$. Given a phase space $V \in \mathrm{Vect}_{\mathbb{C}}$, we can form the symmetric tensor algebra on it, SV: this is the "pre-Fock space", whose Hilbert space completion is the Hilbert space of the <u>quantum</u> harmonic oscillator. E.g. if $V = \mathbb{C}$, $SV = \mathbb{C}[z]$. If $V = \mathbb{C}^n$, $SV = \mathbb{C}[z_1, \ldots, z_n]$.

There's another route:

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, "S \mapsto \mathbb{C}S"] \arrow[d, "S \mapsto [S]"'] & \mathrm{Vect}_{\mathbb{C}} \arrow[d, "V \mapsto SV"] \\
\mathrm{CommMon} \arrow[r, "M \mapsto \mathbb{C}M"'] \arrow[ur, phantom, "\overset{\alpha}{\nearrow}"] & \mathrm{CommAlg}_{\mathbb{C}}
\end{tikzcd}$$

or "$V \to [V]$" ?

Here $[S]$ is the free commutative monoid on S: e.g. if $S = \{z_1, \ldots, z_n\}$ then $[S]$ is the comm. monoid of <u>monomials</u> $z_1^{p_1} \cdots z_n^{p_n}$. Given a comm. monoid M, $\mathbb{C}M$ consists of all formal linear combinations of elts. of M, & this is a commutative algebra. This square commutes up to natural isomorphism:

$$\alpha_S : \mathbb{C}[S] \longrightarrow [\mathbb{C}S]$$

i.e. "$\mathbb{C}$ commutes with $[\ ]$."

E.g.

$$\alpha_S^{-1}\left((z_1 + 2i z_2)(z_3 - 7z_4)\right) = z_1 z_3 - 7 z_1 z_4 + 2i z_2 z_3 - 14 i z_2 z_4 .$$

α_S^{-1} is just the distributive law in action! (it lets us write products of sums as a sum of products)

So, whenever we have a commutative square of <u>free</u> functors, people call it a distributive law.

Physically, if S is a set of "vibrational modes" or "types of particle", the free comm monoid $[S]$ is the set of "collections of particles" – e.g. $z_1^{p_1} \cdots z_n^{p_n}$ represents a collection with p_i particles of type i. These monomials form a basis for the polynomials, so physically these states form a basis of Fock space.

We would like to generalize this in a way that is not specific to $\mathbb{C}$. Let's replace $\mathbb{C}$ by any commutative rig R:

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, "S \mapsto RS"] \arrow[d, "S \mapsto [S]"'] & R\text{-Mod} \arrow[d, "V \mapsto [V]"] \\
\mathrm{CommMon} \arrow[r, "M \mapsto RM"'] & \mathrm{Comm}\ R\text{-Alg}
\end{tikzcd}$$

R-modules & commutative R-algebras are defined for comm. rigs just as for comm. rings

(Note: if your algebra book defines an R module as an abelian group, replace group by monoid – the negatives are superfluous even in the ring case!)

Again, the diagram commutes up to natural isomorphism.

Let's try $R = \mathbb{N}$, the free comm. rig on no generators:

$$
\begin{tikzcd}
\mathrm{Set} \arrow[r, "\text{taking formal sums}"] \arrow[d, "\text{taking formal products}"'] & \mathrm{Comm\,Mon} \ (= \mathbb{N}\text{-}\mathrm{Mod}) \arrow[d, "\text{taking formal products}"] \\
\mathrm{Comm\,Mon} \arrow[r, "\text{taking formal sums}"'] & \mathrm{Comm\,Rig} \ (= \mathrm{Comm}\ \mathbb{N}\text{-}\mathrm{Alg})
\end{tikzcd}
$$

Another famous example: $R = \mathbb{Z}$, the free commutative ring on no generators:

$$
\begin{tikzcd}
\mathrm{Set} \arrow[r, "\text{taking formal sums \& \underline{differences}}"] \arrow[d, "\text{taking formal products}"'] & \mathrm{AbGrp} \arrow[d, "\text{taking formal sums \& differences of formal products}"] \\
\mathrm{Comm\,Mon} \arrow[r, "\text{taking formal sums \& differences}"'] & \mathrm{Comm\,Ring}
\end{tikzcd}
$$

Let's do an example.

A violin string has vibrational modes forming the set

$$\{ \sin x, \sin 2x, \sin 3x, \dots \} \cong \mathbb{N}^+$$

if our string has length π. The frequency of the vibration of the nth mode is proportional to n, depending on tension, mass density of the string. Let's just say $\nu_n = n$ for simplicity. In QM we learn that energy $= \hbar \cdot$ frequency so we can also say $E_n = n$

energy of nth "type of particle."

We'll count the number of states having energy E for the quantized string.