"Violin String Theory":

Start out with an infinitely long string. Its motion satisfies the wave equation:

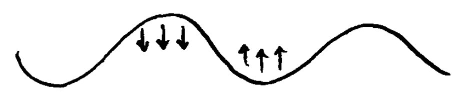
$$\varphi : \mathbb{R}^2 \longrightarrow \mathbb{R}$$

$(t, x) \in \mathbb{R}^2$ — t: time, x: position

$\mathbb{R}$: height of string

$$\boxed{\frac{\partial \varphi}{\partial t^2} = \frac{\partial^2 \varphi}{\partial x^2}}$$

Acceleration — Curvature



curvature $\frac{\partial^2 \varphi}{\partial x^2}$

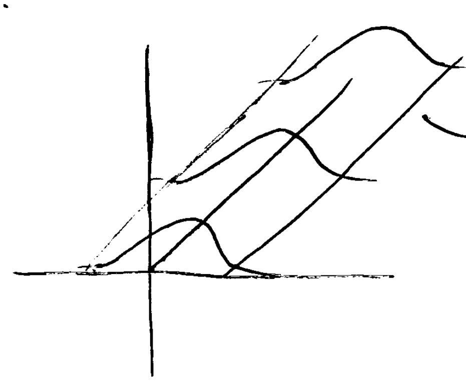
causes acceleration $\frac{\partial^2 \varphi}{\partial t^2}$

This has some obvious solutions:

$$\varphi(t,x) = f(t-x)$$

$$\begin{tikzcd}
& f(t-x) \arrow[dl, "\frac{\partial}{\partial t}"'] \arrow[dr, "\frac{\partial}{\partial x}"] & \\
f'(t-x) \arrow[dr, "\frac{\partial}{\partial t}"'] & & -f'(t-x) \arrow[dl, "\frac{\partial}{\partial x}"] \\
& f''(t-x) &
\end{tikzcd}$$

Solutions look like:



constant along lines $t - x = \text{const}$

Right-moving waves!

We also have left-moving waves:

$$\varphi(t,x) = f(t+x)$$

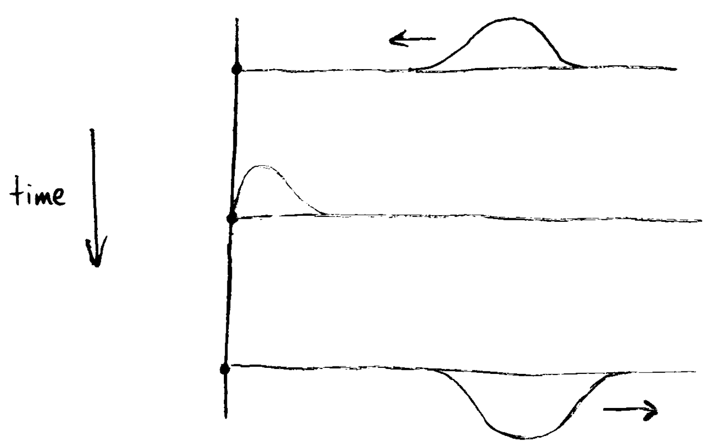
<u>Thm</u>: Every smooth solution of the wave equation on $\mathbb{R}\times\mathbb{R}$ is of the form

$$\varphi(t,x) = f(t+x) + g(t-x)$$

for some $f, g \in C^\infty(\mathbb{R})$.

(<u>Proof</u>: Exercise!)

Suppose we want a solution of the wave equation on $\mathbb{R}\times[0,\infty)$ such that $\varphi(t,0)=0$ for all $t\in\mathbb{R}$.



We can find such solutions by taking:

$$\varphi(t,x) = f(t+x) - f(t-x)$$

This satisfies the wave equation on $\mathbb{R}\times\mathbb{R}$ and has $\varphi(t,-x) = -\varphi(t,x)$ so $\varphi(t,0)=0$ & φ satisfies wave equation on $\mathbb{R}\times[0,\infty)$.

Previous theorem gives:

<u>Cor:</u> Every smooth soln. of the wave eq. on $\mathbb{R} \times [0,\infty)$ with $\varphi(t,0)=0$ is of the form

$$\varphi(t,x) = f(t+x) - f(t-x)$$

for some $f \in C^\infty(\mathbb{R})$.

(Proof: The space of odd functions on $\mathbb{R}$ is isomorphic to the space of function on $[0,\infty)$ that vanish at 0.)

We can also get some nice complex-valued solutions of this type, taking

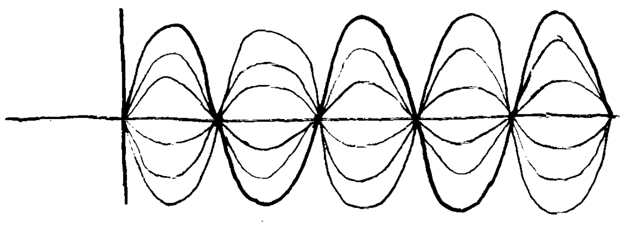
$$f(t) = e^{i\omega t} \qquad t \in \mathbb{R}$$

The real & imaginary parts of a complex solution will be <u>real</u> solutions.

$$\begin{aligned} \varphi(t,x) &= e^{i\omega(t+x)} - e^{i\omega(t-x)} \\ &= e^{i\omega t}(e^{i\omega x} - e^{-i\omega x}) \\ &= 2i\, e^{i\omega t} \sin \omega x \end{aligned}$$

So
$$\begin{aligned} \mathrm{Re}\, \varphi(t,x) &= -2 \sin \omega t \sin \omega x \\ \mathrm{Im}\, \varphi(t,x) &= 2 \cos \omega t \sin \omega x \end{aligned}$$

So we get ~~&~~ something like



These are standing waves with nodes at points where $\sin \omega x = 0$, i.e.

$$x = k\frac{\pi}{\omega} \qquad k = 0, 1, 2, \ldots$$

Now suppose we want standing waves that have nodes at $x = 0$ & $x = \pi$ i.e.

$$\begin{aligned} \varphi(t,0) &= 0 \\ \varphi(t,\pi) &= 0 \end{aligned} \qquad \forall t \in \mathbb{R}$$

Then we need $\pi = k\frac{\pi}{\omega}$, or $\omega = k = 0, 1, 2, 3, \ldots$

(if $\omega < 0$ we just get minus one of these solutions)

This gives solutions:

$$\varphi(t,x) = 2i\, e^{ikt} \sin kx$$

or real solutions:

$$\cos kt \cdot \sin kx \quad \& \quad \sin kt \cdot \sin kx.$$

These are nonzero for $k \in \mathbb{N}^+$

<u>Thm</u>: The space V of smooth real solutions of the wave equation on $\mathbb{R}\times[0,\pi]$ with $\varphi(t,0)=\varphi(t,\pi)=0$ has $\{\underbrace{\cos kt \sin kx}_{=e_k},\ \underbrace{\sin kt \sin kx}_{=f_k}\}$ as a topological basis, i.e. every linear comb.

$\sum_{k\in\mathbb{N}^+} a_k e_k + b_k f_k$ where $a_k, b_k \to 0$ faster than $\frac{1}{k^p}$ for any power of p converges to a point in V (with its C^∞ topology), and every point of V is of this form for a unique choice of $\{a_k\},\{b_k\}$.

$$\boxed{\begin{array}{l} f_i \to f \text{ in } C^\infty \text{ top.} \\ \text{if} \\ \sup_{t,x} |\partial_x^n \partial_x^m (f_i - f)| \to 0 \\ \quad \forall n,m \end{array}}$$

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topological direct sum = closure of algebraic direct sum.

Note that $V = \bigoplus_{k\in\mathbb{N}^+} V_k$ where V_k has basis e_k, f_k.

We have linear operators describing time evolution:

$$U(t): V \longrightarrow V \qquad \forall t\in\mathbb{R}$$

given by:

$$(U(t)\varphi)(s,x) = \varphi(t+s,x)$$

&

$$U(t): V_k \longrightarrow V_k$$

So: our physics problem here is a direct sum of little physics problems — harmonic oscillators with different frequencies!

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We saw that the space of smooth solns of the wave equation on $\mathbb{R} \times [0, \pi]$ with Dirichlet boundary conditions:

$$\varphi(t,0) = \varphi(t,\pi) = 0 \quad \forall t$$

has a topological basis given by

$$e_k = \cos kt \ \sin kx$$

$$f_k = \sin kt \ \sin kx$$

Let

$$V_k = \{a e_k + b f_k \ : \ a, b \in \mathbb{R}\}$$

and now let's work purely algebraically & redefine V to be the space of finite linear combinations of e_k, f_k (so we're taking a smaller space, but it's dense in the original one) so

$$V = \bigoplus_{k \in \mathbb{N}^+} V_k$$

where now this is an algebraic direct sum.

Time evolution

$$U(t) : V \longrightarrow V$$

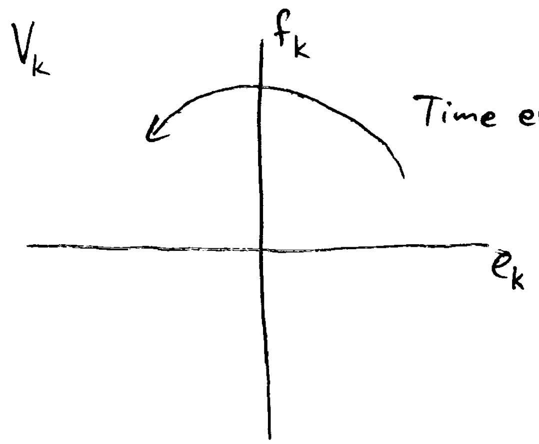
preserves each subspace V_k:

$$U(t) : V_k \longrightarrow V_k$$

$$
\begin{aligned}
U(s)\, e_k &= \cos k(t+s) \sin kx \\
&= (\cos ks \cos kt - \sin ks \sin kt) \sin kx \\
&= \cos ks \; e_k - \sin ks \; f_k \\
U(s)\, f_k &= \sin k(t+s) \sin kx \\
&= (\sin ks \cos kt + \cos ks \sin kt) \sin kx \\
&= \sin ks \; e_k + \cos ks \; f_k
\end{aligned}
$$

So

$$
U(s)\Big|_{V_k} = \begin{pmatrix} \cos ks & -\sin ks \\ \sin ks & \cos ks \end{pmatrix}
$$



Time evolution is rotation at rate $k \in \mathbb{N}^+$

Just as we did with the harmonic oscillator, Let's make V_k into a 1-dim complex vector space with

$$
\begin{aligned}
i e_k &= f_k \\
i f_k &= -e_k
\end{aligned}
$$

Then we get

$$U(s)\Big|_{V_k} = e^{iks}$$

Now, let's <u>QUANTIZE</u> the string!

Recall:

$$\{e_k\}_{k\in\mathbb{N}^+} \longmapsto \mathbb{C}\{e_k\} = V$$

$$\begin{tikzcd}
\mathrm{Set} \arrow[r, "\mathbb{C}-"] \arrow[d, "{[-]}"'] & \mathrm{Vect}_{\mathbb{C}} \arrow[d, "S"] \\
\mathrm{CommMon} \arrow[r, "\mathbb{C}-"] & \mathrm{CommAlg}_{\mathbb{C}}
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[d, mapsto, "\text{“quantization”}"] \\
SV
\end{tikzcd}$$

V ← classical phase space = space of solns of wave eq.

SV ← quantum state space = "pre-Fock space of V"

need to complete this to get the Fock space.

V has basis of e_k's so SV has a basis of monomials:

$$e_1^{p_1} e_2^{p_2} \cdots e_k^{p_k}$$

where $k \in \mathbb{N}^+$ is arbitrary & $p_i \in \mathbb{N}$. Physically this is the state with "p_i particles of type i". The monomial 1 is also called the <u>vacuum</u>.

To quantize <u>time evolution</u> for the string, we use the fact that $S: \mathrm{Vect}_{\mathbb{C}} \longrightarrow \mathrm{CommAlg}_{\mathbb{C}}$ is a functor: it not only sends objects (vector spaces) to objects (comm. algs.), but morphisms (linear maps) to morphisms (algebra homomorphisms)! So, time evolution for the classical string:

$$U(t): V \longrightarrow V$$

gets sent to

$$SU(t) : SV \longrightarrow SV$$

which is time evolution for the quantum string!

Here's how it goes: Given a linear operator

$$f: L \longrightarrow M$$

we have

$$Sf : SL \longrightarrow SM$$

$$l_1 \cdots l_n \longmapsto f(l_1) \cdots f(l_n) \qquad l_i \in L$$

& Sf is linear, hence determined on all of SL — in fact a homomorphism.

In particular, $SU(t) : SV \longrightarrow SV$ is given by:

$$SU(t)(e_1^{p_1} e_2^{p_2} \cdots e_k^{p_k})$$
$$= (U(t)e_1)^{p_1} (U(t)e_2)^{p_2} \cdots (U(t)e_k)^{p_k}$$
$$= (e^{it} e_1)^{p} (e^{i2t} e_2)^{p_2} \cdots (e^{ikt} e_k)^{p_k}$$

$$= e^{it(1p_1+2p_2+\cdots+kp_k)}\, e_1^{p_1} e_2^{p_2} \cdots e_k^{p_k}$$

In fact

$$SU(t) = e^{itH}$$

where $H: SV \longrightarrow SV$, the Hamiltonian for the quantized string, is:

$$H\, e_1^{p_1} \cdots e_k^{p_k} = (1p_1 + 2p_2 + \cdots + kp_k)\, e_1^{p_1} \cdots e_k^{p_k}.$$

Moral: the state $e_1^{p_1} \cdots e_k^{p_k}$ is an energy eigenstate with energy $E = 1p_1 + \cdots + kp_k$. So: the ith type of particle has energy i!!

Now: How many eigenstates are there with energy $n \in \mathbb{N}$? We'll call this number $P(n)$, the nth partition number

$P(0) = 1$	(vacuum) $0 =$
$P(1) = 1$	$1 = 1$
$P(2) = 2$	$2 = 1+1,\ 2$
$P(3) = 3$	$3 = 1+1+1,\ 1+2,\ 3$
$P(4) = 4$	$4 = 1+1+1+1,\ 1+1+2,\ 1+3,\ 2+2,\ 4$
$P(5) = 7$	$5 = 1+1+1+1+1,\ 1+1+1+2,\ 1+1+3,\ 1+2+2,\ 1+4,\ 2+3,\ 5$

$\vdots$

Ramanujan studied these

The generating function for the $P(n)$ is, by the homework:

$$\sum_{n\in\mathbb{N}} P(n)\, z^n = \prod_{k\in\mathbb{N}^+} \frac{1}{1-z^k},$$

the <u>partition function</u> of the string.

Ramanujan studied these & showed lots of bizarre things, e.g.:

$$P(5m+4) \equiv 0 \bmod 5$$

How? He defined

$$\phi(z) = \prod_{k=1}^{\infty} (1-z^k)$$

& he showed:

$$\sum_{m\geq 0} P(5m+4)\, z^m = 5 \underbrace{\frac{\phi(z^5)^5}{\phi(z)^6}}$$

where this $= \sum a_n z^n$ w. $a_n \in \mathbb{N}$!

These curiosities can be more systematically obtained via string theory!