

We've seen generating functions in two completely different roles:

- 1) The generating function of any structure type is an element of  $\mathbb{C}[[z]]$ , which contains the Fock space of a harmonic oscillator with 1 degree of freedom as its finite-norm elements.

So: you can think of structure types as categorized states of the harmonic oscillator with 1 degree of freedom. We will generalize this later to many degrees of freedom, getting "structure types on  $n$ -tuples of sets":

$$F: \text{FinSet}_0^n \rightarrow \text{Set}$$

- 2) We can also use generating functions to count the number of states of energy  $n$  of some system, e.g. a harmonic oscillator with many degrees of freedom.

Example: Suppose you have a classical harmonic oscillator that has modes with

vibrational frequencies in  $S \subseteq \mathbb{N}^+$ . If we quantize it, we get  $p_n$  states of energy  $n$  where  $p_n = \#$  of ways of writing  $n$  as a sum of elts of  $S$ .

$$\sum_{n \in \mathbb{N}} p_n z^n = \prod_{k \in S} \frac{1}{1-z^k}$$

E.g.:

a) harmonic oscillator w. one degree of freedom:

$$S = \{1\}$$

b) open string with Dirichlet boundary conditions:

$$S = \mathbb{N}^+.$$

Q: What is the physical meaning of the second use of ~~the 2nd kind~~ generating functions?

It's called the Zustandssumme! A "sum over states" or "partition function".

Basic Fact of Statistical Mechanics:

Suppose a physical system has states  $x \in X$  with energies  $H(x) \in \mathbb{R}$ :

$$H: X \rightarrow \mathbb{R}$$

At temperature  $T$ , the probability of finding the system in state  $x$  is proportional to

$$e^{-H(x)/kT}$$

where  $k$ , Boltzmann's constant is

$$k = 1.3807 \cdot 10^{-23} \text{ J/K.}$$

Since mathematicians can't remember numbers, they set

$$\beta = \frac{1}{kT}$$

& say relative probability is

$$e^{-\beta H(x)}$$

**Hot**  $\Rightarrow T$  big  $\Rightarrow \beta$  small  $\Rightarrow$  this exponential decays slowly as a fn. of energy  $H(x)$   
 $\Rightarrow$  high-energy states are almost as probable as lower energy ones.

**Cold**  $\Rightarrow$  high-energy states are much less probable than low-energy ones.

"Relative probability":

$$\frac{\text{prob. of state } x}{\text{prob of state } x'} = \frac{e^{-\beta H(x)}}{e^{-\beta H(x')}} = e^{-\beta(H(x) - H(x'))}$$

So to get actual probabilities, we have to normalize using the sum of relative probabilities of all states (the partition fn.):

$$\text{prob of state } x = \frac{e^{-\beta H(x)}}{\sum_{x \in X} e^{-\beta H(x)}} \quad \leftarrow \text{partition function.}$$

Example: suppose  $X = \mathbb{N}$  and  $H(n) = n$   
 (like the harmonic oscillator - one state of energy  $n$  for each  $n \in \mathbb{N}$ ). Then the partition function is

$$Z(\beta) = \sum_{n \in \mathbb{N}} e^{-\beta n} = \frac{1}{1 - e^{-\beta}}$$

Now if we let

$$\boxed{e^{-\beta} = z}$$

this becomes

$$\sum_{n \in \mathbb{N}} z^n = \frac{1}{1 - z}$$

which is just the generating function of "being a totally ordered set" - i.e.  $\sum_{n \in \mathbb{N}} p_n z^n =$   
 # of ways of writing  $n$  as a sum of 1's.

Example: Partition fn for the open string is

$$\prod_{k \in \mathbb{N}^+} \frac{1}{1 - e^{-k\beta}} = \prod_{k \in \mathbb{N}^+} \frac{1}{1 - z^k}$$

which is the generating function that counts ways of writing  $n$  as a sum of positive natural numbers.

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## CLASSICAL VERSUS QUANTUM THERMODYNAMICS

### Classical

Pure states are points  $x$  in some space (e.g. manifold)  $X$ .

Observables are functions

$$O: X \rightarrow \mathbb{R}$$

The value of  $O$  in state  $x$  is  $O(x)$ .

### Quantum

Pure states are rays (i.e. 1d subspaces) in some Hilbert space  $K$ , or equivalently, equivalence classes of unit vectors  $\psi \in K$  where  $\psi \sim \psi'$  if  $\psi' = c\psi$  for  $|c| = 1$   $c \in \mathbb{C}$  (Projective Hilbert space  $PK$ )

Observables are self-adjoint operators  $O$  (densely defined) on  $K$ . The expected value of  $O$  in the state is  $\langle \psi, O\psi \rangle$

(Classical)

Mixed states are probability measures on  $X$ , say  $\mu$ :

- $\mu \geq 0$   
( $f \geq 0 \Rightarrow \int f d\mu \geq 0$ )
- $\int d\mu = 1$

The expected value of  $O$  in the mixed state  $\mu$  is:

$$\langle O \rangle = \int O(x) d\mu(x)$$

(Note:  $C(X)^* = M(X)$ )

if  $X$  is locally compact Hausdorff space

$C(X)$  = bded continuous fns.  
 $M(X)$  = regular Borel measures)



(Quantum)

Mixed states are density matrices i.e linear operators  $D:K \rightarrow K$  s.t.

- $D \geq 0$  (i.e.  $D$  self-adjoint, its spectrum lies in  $[0, \infty)$ .)
- $\text{tr}(D) = 1$  (where  $\text{tr}(D) = \sum_i \langle e_i, D e_i \rangle$  for any o.n. basis  $e_i$  of  $K$  assuming absolute convergence.)

The expected value of  $O$  in the mixed state  $D$  is

$$\langle O \rangle = \text{tr}(OD)$$

(Note:  $L(K)^* = L'(K)$ )

if  $K$  Hilbert space

$L(K)$  = bded operators  
 $L'(K)$  = trace-class operators



(Classical)

The pure state  $x$  gives a special mixed state  $\delta_x$ , the Dirac delta measure at  $x$ .

Consistency check:

$$\int O(y) \delta_x(y) = O(x)$$

Given a Hamiltonian

$$H: X \rightarrow \mathbb{R}$$

the Gibbs state is the mixed state describing thermal equilibrium at temperature  $T$ . It is:

$$\frac{e^{-\beta H(x)} dx}{\int e^{-\beta H(x)} dx}$$

where  $\beta = \frac{1}{kT}$  and  $dx$  is some positive measure on  $X$  that we need in order to do statistical mechanics! Note: this only makes sense if

$$Z(\beta) = \int e^{-\beta H(x)} dx \text{ converges}$$

(Quantum)

The pure state  $\psi$  gives a special mixed state  $P_\psi$ , the projection operator

$$P_\psi \phi = \langle \psi, \phi \rangle \psi$$

consistency check:

$$\text{tr}(OP_\psi) = \langle \psi, O\psi \rangle$$

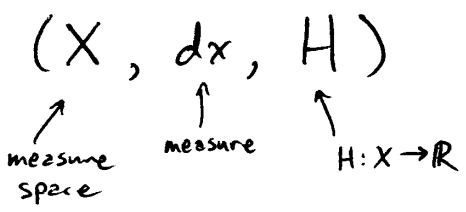
Given a Hamiltonian  $H$  on  $K$ , the Gibbs state is

$$\frac{e^{-\beta H}}{\text{tr}(e^{-\beta H})}$$

Note: this only makes sense if  $Z(\beta) = \text{tr}(e^{-\beta H})$  converges.

(Classical)

There's a category  $\mathcal{C}Therm$  of classical thermodynamic systems: objects are



$\mathcal{C}Therm$  is a 2-rig with

- $+$  : disjoint union
- $\times$  : Cartesian product (need to add energies)

The partition function gives a rig homo:

$$Z: \mathcal{C}Therm \longrightarrow C(0, \infty)$$

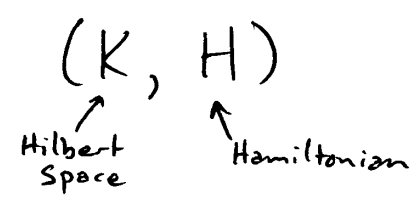
(comm rig)

So

$$\begin{aligned} Z(X+Y) &= Z(X) + Z(Y) \\ Z(X \times Y) &= Z(X)Z(Y) \end{aligned}$$

(Quantum)

There's a category  $\mathcal{Q}Therm$  of quantum thermodynamic systems: objects are



$\mathcal{Q}Therm$  is a 2-rig with

- $\oplus$  - direct sum
- $\otimes$  - tensor product

We get a rig homo:

$$Z: \mathcal{Q}Therm \longrightarrow C(0, \infty)$$

(comm rig)

So

$$\begin{aligned} Z(X \oplus Y) &= Z(X) + Z(Y) \\ Z(X \otimes Y) &= Z(X)Z(Y) \end{aligned}$$



